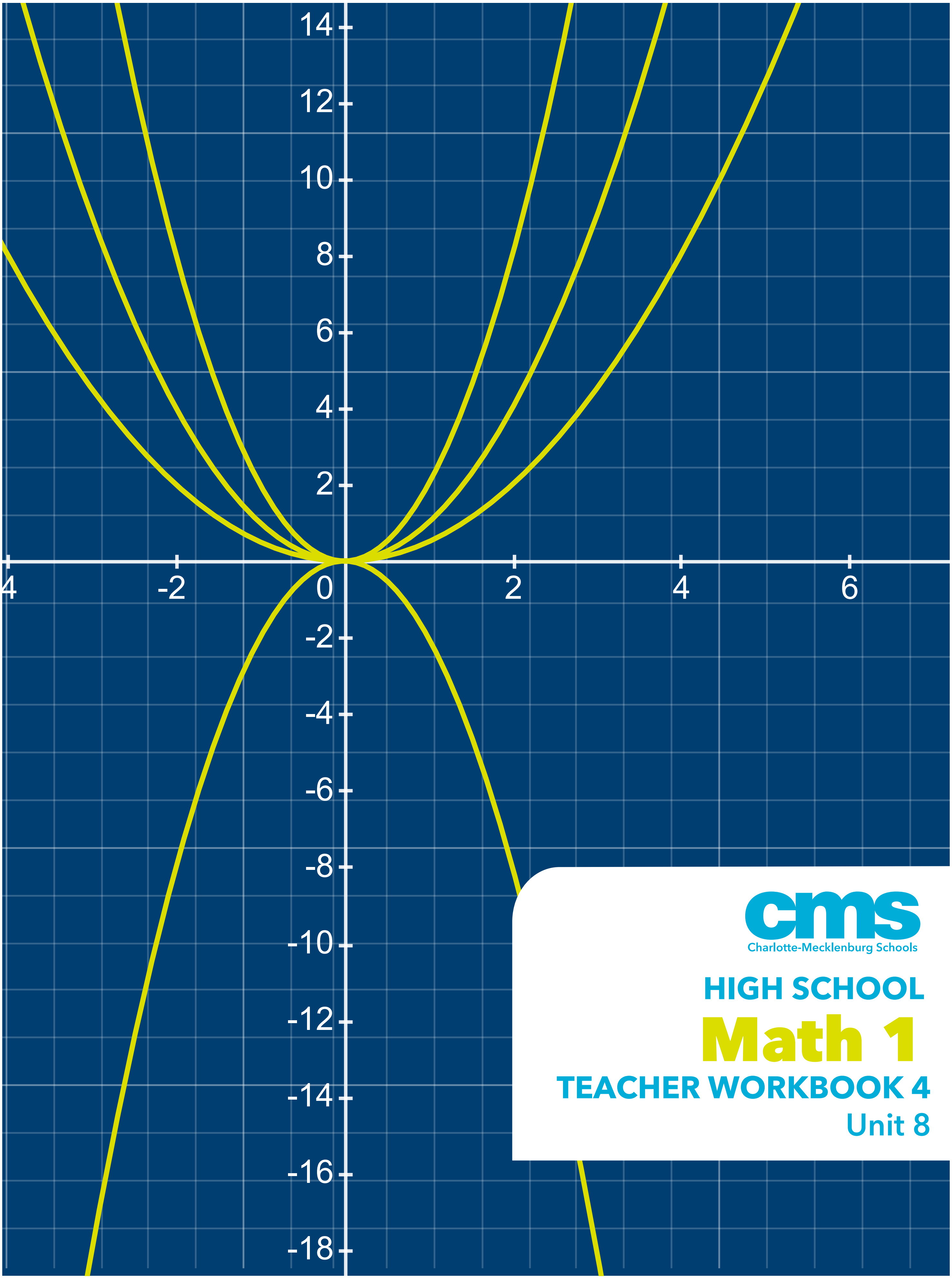


Charlotte-Mecklenburg Schools

HIGH SCHOOL

Math 1

TEACHER WORKBOOK 4

Unit 8

Table of Contents

Unit 8: Sequences and Functions

This unit provides an opportunity to revisit representations of functions (including graphs, tables, and expressions) and function types using the example of a sequence as a particular type of function. Through many concrete examples, students learn to identify geometric and arithmetic sequences. They see them as examples of the exponential and linear functions they learned about earlier in the year, defined by a subset of the integers.

Students begin with an invitation to describe sequences with informal language.They write out the terms of sequences arising from mathematical situations, in addition to interpreting and creating tables and graphs about the given relationship. In Lesson 1, they solve a classic puzzle that involves recursive reasoning, and find the pattern in the number of moves it takes to solve each successive version of the puzzle. In Lessons 2 and 3, they learn about geometric and then arithmetic sequences.

In Lesson 4, students learn that sequences are a type of function in which the input variable is the position and the output variable is the term at that position. They learn to interpret and then write their own definitions for sequences recursively using function notation. Lesson 5 introduces subscript notation as an alternative way to write recursive definitions.

In Lesson 6, expressions for the n^{th} term of a sequence are built up through expressing regularity in repeated reasoning (MP8), building on students' prior experiences studying linear and exponential functions. For example, the geometric sequence 6, 18, 54, 162, . . . could be written as $6,\ 6 \cdot 3,\ 6 \cdot 3 \cdot 3,\ 6 \cdot 3 \cdot 3 \cdot 3,\ \ldots$, which makes it clearer to see that the n^{th} term can be defined by $f(n) = 6 \cdot 3^{n-1}$, assuming we start at $f(1) = 6$.

In Lesson 7, students use sequences to model situations represented in different ways (MP4). This isn't meant to be full-blown modeling but touches on some practices that must be attended to while modeling, such as choosing a good model, identifying an appropriate domain, or expressing numbers with an appropriate level of precision given the situation. Students also recognize that a sequence is an appropriate type of function to use as a model for these situations since the domain of each is a subset of the integers.

Lessons 8 and 9 are modeling lessons. In addition to the prompts that were included in Units 2, 4, and 6, two additional prompts are provided here. In Modeling Prompt #7, students plan a vacation while being careful to stay within the constraints of a budget. In Modeling Prompt #8, students plan a fundraising concert, this time making decisions based on revenue as well as cost. Lesson 10 occurs after administering the Unit 8 assessment and includes post-assessment activities.

A note on standards: Because the new work with arithmetic and geometric sequences builds on students' prior understanding of linear and exponential functions, many activities are tagged as both "building on" and "building towards" NC.M1.F-BF.1a (*Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table*).

Instructional Routines

Aspects of Mathematical Modeling: Lessons 7, 8 & 9

Co-Craft Questions (MLR5): Lessons 2, 7

Collect and Display (MLR2): Lessons 3, 5

Compare and Connect (MLR7): Lesson 7

Discussion Supports (MLR8): Lessons 4, 5

Notice and Wonder: Lessons 2, 5, 6

Poll the Class: Lesson 7

Stronger and Clearer Each Time (MLR1): Lesson 6

Take Turns: Lessons 5, 6

Which One Doesn't Belong?: Lesson 6

Lesson 1: Investigating Sequences

PREPARATION

Lesson Goals	Learning Targets
• Comprehend the term "sequence" (in written and spoken language) as a list of numbers. • Describe (orally) a recursive rule for identifying the next term of a simple sequence. • Generate a sequence that arises from a mathematical context.	• I can give an example of a sequence. • When I see a sequence with a pattern, I can describe a rule for finding the next term of the sequence.

Lesson Narrative

The purpose of this lesson is for students to work with sequences and describe them recursively in an informal way. A **sequence** is defined here as a list of numbers while a **term (of a sequence)** is one of the numbers in the list.

Solving the puzzles provides students opportunities to express regularity in repeated reasoning (MP8) when they informally state a recursive rule for generating the next term in a sequence from the previous term.

In what ways will you encourage students to persevere in this lesson?

Focus and Coherence

Building On	Building Towards
NC.8.F.4: Analyze functions that model linear relationships. • Understand that a linear relationship can be generalized by $y = mx + b$. • Write an equation in slope-intercept form to model a linear relationship by determining the rate of change and the initial value, given at least two (x, y) values or a graph. • Construct a graph of a linear relationship given an equation in slope-intercept form. • Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of the slope and y-intercept of its graph or a table of values.	**NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). **NC.M1.F-BF.2:** Translate between explicit and recursive forms of arithmetic and geometric sequences and use both to model situations.

Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*25 minutes*)
  - Jumping Checkers Puzzle (Applet preferred): https://bit.ly/JumpingCheckers
    - Tokens: If not using the digital applet, each group needs at least 3 tokens each of 2 different colors. These could be actual checkers, counting chips, pennies and nickels, or any other appropriate tokens.
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U8.L1 Cool-down (print 1 copy per student)

LESSON

Bridge (*Optional, 5 minutes*)

Building On: NC.8.F.4

This bridge is designed to activate students' knowledge of linear relationships before they investigate arithmetic sequences in upcoming lessons. Students will likely approach this problem by figuring out the amount of time per student, which relates to the previously studied concept of slope and the upcoming concept of a common difference.

Student Task Statement

Clare Trujillo is graduating from West Mecklenburg High School this year! In her class of 368 graduates, she is number 323 in line. The awarding of diplomas starts at 1:00 p.m. with Andre Abney as the first graduate. Tyler Graham is the 120th graduate to walk across the stage, at 1:12 p.m. Assuming every graduate takes about the same amount of time, at what time should Clare expect to walk across the stage?

DO THE MATH	PLANNING NOTES

Warm-up: What's Next? (*5 minutes*)

Building Towards: NC.M1.F-BF.1a

The purpose of this warm-up is for students to generate a list of numbers based on a rule. There is no need to identify any specific vocabulary such as "sequence," "term," or "arithmetic" at this time.

In the next activity, students will generate a list of numbers from a puzzle and then describe the pattern in the list.

Avoid demonstrating a possible sequence of numbers for the whole class in order to reinforce the expectation that students are responsible for making sense of the activities.

Step 1

- Ask students to arrange themselves in pairs or use visibly random grouping. Students will remain in these groups for the entire lesson.
- Tell students there are many possible answers. Suggest partners agree on a starting number and then after brief quiet work time, ask students to compare their responses to their partner's.

Monitoring Tip: As students are working, monitor for students who have generated a successful sequence, representing at least two different starting numbers, and let them know that they may be asked to share later. Include at least one student who does not typically volunteer.

***Advancing Student Thinking*:** If a student has trouble getting started, ask them to first pick a number. Once they've picked a number, ask what the next number would be, according to the rule. Then invite them to continue.

Some students may interpret "1 less than twice a number" as $1-2x$ (where x is the number). Ask these students questions like:

- "What's 1 less than 5?" (4)
- "What's 1 less than 5 times 2?" (9)
- "What's 1 less than 100?" (99)
- "What's 1 less than 100 times 2?" (199)

Write some of these statements on the board along with their translation into mathematical expressions. For example, write "1 less than 5 times 2" next to $5 \cdot 2 - 1$.

Student Task Statement

Here is a rule for making a list of numbers:

Each number is 1 less than twice the previous number.

Pick a number to start with, then follow the rule to build a list of five numbers.

Step 2

Invite previously identified students to present their lists. Ask the other students to verify their calculations.

DO THE MATH	PLANNING NOTES

Activity 1: Jumping Checkers Puzzle (*25 minutes*)

Building Towards: NC.M1.F-BF.1a; NC.M1.F-BF.2

The purpose of this activity is to define "sequence." Students look for a pattern in the sequence generated by listing the number of moves needed to solve the puzzle for different numbers of checkers and then describe the pattern informally (MP8).

This activity works best when each group has access to manipulatives or devices that can run the GeoGebra applet because students will benefit from seeing the relationship in a dynamic way: https://bit.ly/JumpingCheckers. If students don't have individual access, projecting the applet during the launch would be helpful.

Step 1

- Set up either a physical puzzle with two checkers (or discs) on each side, or display the digital version to display two red checkers and two blue checkers.
- Display this or a similar table for all to see throughout the discussion:

Number of checkers on each side	1	2	3	4	5	6	7
Number of moves							

- Ask students to read the rules of the puzzle. Next, invite students to name what moves are possible (move the rightmost blue checker one space to the right; move the leftmost red checker one space to the left) and which are not allowed (move the leftmost blue checker to the middle open space).
- Complete the puzzle for two checkers on each side, asking students to suggest moves to complete the puzzle. Fill in the table for the number of moves needed (8). Tell students that now it is their turn to figure out the number of moves needed for different numbers of checkers. Distribute objects with which to experiment with the puzzle. Alternatively, help students access the digital applet.

Student Task Statement

Some checkers are lined up, with blue on one side (left), red on the other (right), with one empty space between them. A "move" in this checker game pushes any checker forward one space or jumps over any one checker of the other color. Jumping the same color is not allowed, moving backwards is not allowed, and two checkers cannot occupy the same space.

Complete the puzzle by switching the colors completely: ending up with blue on the right, red on the left, with one empty space between them.

1. Using 1 checker on each side, complete the puzzle. What is the smallest number of moves needed?
2. Using 3 checkers on each side, complete the puzzle. What is the smallest number of moves needed?
3. Looking at the table so far, guess the number of moves needed if there are 4 checkers on each side. Then test your guess by solving the puzzle using 4 checkers on each side.
4. How many moves do you think it will take to complete a puzzle with 7 checkers on each side?

Step 2

- Facilitate a whole-class discussion. The goal of the discussion is to define "sequence" and "term (of a sequence)."
  - Select students to share the number of moves they found for 1, 3, and 4 checkers on each side. The bottom row of the table should now have the numbers 3, 8, 15, and 24 filled in. If some students were not successful solving the puzzle for 3 or 4 checkers on a side, ask for a volunteer to demonstrate or demonstrate it yourself.

- Conclude the discussion by telling students that in mathematics, we often call a list of numbers a “sequence.” The list 3, 8, 15, 24 is an example of a sequence. A specific number in the list is called a “term” of the sequence. Ask students how they would describe the rule for the next term in this sequence.
  - After a brief quiet think time, select two or three students to share their thinking and write down any notation they come up with to describe the recursive rule, such as “first add 5, and then keep adding the next odd number.” There is no need to introduce formal notation or discuss a specific rule for finding term n at this time, but if students suggest these, welcome their explanations (time permitting).

RESPONSIVE STRATEGY

Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of: term and sequence.

Supports accessibility for: Conceptual processing; Language

DO THE MATH	PLANNING NOTES

Lesson Debrief (*5 minutes*)

The purpose of this lesson is to introduce students to the idea of a sequence of numbers and to begin to describe sequences orally.

	PLANNING NOTES
Ask students to come up with a sequence of five numbers that follows some rule and write it down on a piece of paper. After a brief quiet think time, tell students to exchange papers with another student in the class and then try to find the next term in their partner's sequence along with the rule they used. Conclude the lesson by inviting students to share a sequence and rule their partner wrote that they found interesting. Display these for all to see while students share.	

Student Lesson Summary and Glossary

A list of numbers like 3, 5, 7, 9, 11, . . . or 1, 5, 13, 29, 61, . . . is called a **sequence**.

There are many ways to define a sequence, but one way is to describe how each **term** relates to the one before it. For example, the sequence 3, 5, 7, 9, 11, . . . can be described this way: the starting term is 3, then each following term is 2 more than the one before it. The sequence 1, 5, 13, 29, 61, . . . can be described as: the starting term is 1, then each following term is the sum of 3 and twice the previous term.

Throughout this unit, we will study several types of sequences along with ways to represent them.

Sequence: A list of numbers, possibly going on forever, such as all the odd positive integers arranged in order: 1, 3, 5, 7, ...

Term (of a sequence): One of the numbers in a sequence.

Cool-down: Next? (*5 minutes*)

Building Towards: NC.M1.F-BF.1a; NC.M1.F-BF.2

Cool-down Guidance: More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.

Cool-down

A sequence starts 3, 6, . . .

1. Give a rule the sequence could follow, and list the next three terms.
2. Give a **different** rule the sequence could follow, and list the next three terms.

Student Reflection:

Do you enjoy doing math puzzles? If yes, what do you like about them? If no, why do you think that is the case?

DO THE MATH

INDIVIDUAL STUDENT DATA

SUMMARY DATA

NEXT STEPS

TEACHER REFLECTION

What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?

What do you love most about math? How are you sharing that joy with your students and encouraging them to think about what they love about math?

Practice Problems

1. Here is a rule to make a list of numbers: Each number is the sum of the previous two numbers.

   Start with the numbers 0 and 1, then follow the rule to build a sequence of ten numbers.

2. A sequence starts $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$

   a. Give a rule that the sequence could follow.

   b. Follow your rule to write the next three terms in the sequence.

3. A sequence of numbers follows the rule: multiply the previous number by -2 and add 3. The fourth term in the sequence is -7.

    a. Give the next three terms in the sequence.

    b. Give the three terms that came before -7 in the sequence.

4. A sequence starts 0, 5, . . .

    a. Give a rule the sequence could follow and the next three terms for that rule.

    b. Give a **different** rule the sequence could follow and the next three terms for that rule.

5. Consider the expression $(5+x)(6-x)$.

    a. Is the expression equivalent to x^2+x+30? Explain how you know.

    b. Is the expression $30+x-x^2$ in standard form? Explain how you know.

    (From Unit 7)

6. Explain or show why the product of a sum of two quantities and a difference of the same two quantities, such as $(2x+1)(2x-1)$, has no linear term when written in standard form.

    (From Unit 7)

7. A bank account pays 0.5% monthly interest.

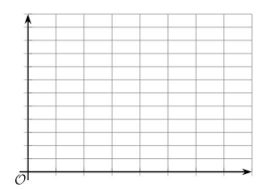
    a. If $500 is put in the account, what will the balance be at the end of one year, assuming no additional deposits or withdrawals are made?

    b. What is the effective annual interest rate?

    c. Is the effective annual interest rate more or less than 6% (the nominal interest rate)?

    (From Unit 6)

8. Kiran bought a smoothie every day for a week. Smoothies cost $3 each. The amount of money he spends, in dollars, is a function of the number of days of buying smoothies.



    a. Sketch a graph of this function. Be sure to label the axes.

    b. Describe the domain and range of this function.

    (From Unit 5)

9. Select **all** the values for r, the correlation coefficient, that indicate a strong, negative relationship for the line of best fit.

    a. 1
    b. -0.97
    c. -0.45
    d. 0.53
    e. 0.9
    f. -0.8
    g. -1

    (From Unit 4)

10. The environmental science club is printing T-shirts for its 15 members. The printing company charges a certain amount for each shirt plus a setup fee of $20.

    If the T-shirt order costs a total of $162.50, how much does the company charge for each shirt?

    (From Unit 3)

Lesson 2: Introducing Geometric Sequences

PREPARATION

Lesson Goals	Learning Targets
• Create tables and graphs to represent geometric sequences. • Determine missing terms in geometric sequences. • Determine the common ratio of a geometric sequence.	• I can create tables and graphs to represent geometric sequences. • I can find missing terms in a geometric sequence. • I can determine the common ratio of a geometric sequence.

Lesson Narrative

The purpose of this lesson is for students to understand what makes a sequence a **geometric sequence** and to begin to connect that idea with their learning about exponential functions in an earlier unit. The lesson also gives students the opportunity to use precise language to describe the relationship between consecutive terms in a sequence (MP6): in particular, how the terms of a geometric sequence grow by the same factor from one term to the next. For example, this is a geometric sequence: 0.5, 2, 8, 32, 128, . . . Each term is 4 times the previous term. Two ways to think about why the sequence is geometric are:

- Each term is multiplied by a factor of 4 to get the next term.
- The ratio of each term to the previous term is 4.

We call 4 the "growth factor" or the **common ratio**. After considering some examples of geometric sequences in the warm-up and finding similarities, students then develop two different sequences to describe continually cutting a piece of paper in half. Using tables and graphs to identify the common ratio of these sequences, geometric sequences are defined. Next, students practice calculating missing terms of geometric sequences by first identifying the common ratio.

Earlier in this course, students studied exponential functions. Some students may see that a geometric sequence is simply an exponential function whose outputs are the terms and whose inputs are the positions of the terms. Later lessons will elaborate on this idea. This lesson invites recall of those ideas with a light touch by referring to, for example, "the size of each piece as a function of the number of cuts" and by using the term "factor" or "common ratio." Students are also asked to describe how graphs representing each quantity in the paper-cutting context are the same and different, and in so doing may recall prior knowledge of the behavior of exponential functions. In the last activity, the connection is made explicit.

A note on language and notation: in earlier grades and units, students may have learned that a ratio is an association between two or more quantities. As students advance in mathematics, "ratio" is typically used as a synonym for "quotient." This expanded use of the word "ratio" comes into play in this lesson with the introduction of the term "common ratio."

Share some ways you see this lesson connecting to previous lessons in this course. What connections will you want to make explicit?

Focus and Coherence

Building On	Building Towards
NC.M1.F-BF.1a: Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).	**NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). **NC.M1.F-BF.2:** Translate between explicit and recursive forms of arithmetic and geometric sequences and use both to model situations.

Agenda, Materials, and Preparation

- **Warm-up** (*10 minutes*)
- **Activity 1** (*15 minutes*)
  - Prepare one pair of scissors and one 8-inch-by-10-inch piece of blank paper to demonstrate the paper cutting described in the task statement.
- **Activity 2** (*10 minutes*)
  - Technology is required for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U8.L2 Cool-down (print 1 copy per student)

LESSON

Warm-up: A Pattern in Lists (*10 minutes*)

Instructional Routine: Notice and Wonder	
Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.1a; NC.M1.F-BF.2

The purpose of this task is to re-introduce growth and decay factors from earlier in this course and to begin using the term “common ratio” when factors are used to describe sequences. Students notice and describe the fact that each sequence is characterized by the same type of relationship between consecutive terms.

When students articulate what they *Notice and Wonder*, they have an opportunity to attend to precision in the language they use to describe what they see (MP6). They might use less formal or precise language at first and then restate their observation with more precise language in order to communicate more clearly.

The last two sequences may present a challenge since the factor is less than 1. The purpose of including these sequences is to encourage students to notice and make use of structure (MP7). If they notice that in the first two sequences, each pair of consecutive terms has the same quotient, they could inspect the quotients in the last sequence. These two sequences also give the opportunity to point out that the term “common ratio” can be used whether the sequence’s factor is a growth or decay factor.

RESPONSIVE STRATEGY

Provide students with a graphic organizer to record what they notice and wonder.

Supports accessibility for: Language; Organization

Step 1

- Ask students to arrange themselves in pairs or use visibly random grouping. Students will remain with these partners for the remainder of the lesson.
- Display the four sequences for all to see.
- Ask students to think about what they notice or wonder about the sequences.
- Give students 1 minute of quiet think time and then 1 minute to discuss the things they notice and wonder with their partner.

Student Task Statement

What do you notice? What do you wonder?

- 40, 120, 360, 1080, 3240
- 2, 8, 32, 128, 512
- 1000, 500, 250, 125, 62.5
- 256, 192, 144, 108, 81

Step 2

- Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the sequence. After all responses have been recorded without commentary or editing, ask students, "Is there anything on this list that you are wondering about now?" Encourage students to respectfully disagree, ask for clarification, or point out contradicting information.
- If the idea of each consecutive term in a sequence growing by the same factor or having a common ratio does not come up during the conversation, ask students to discuss this idea. Students may describe how the "same thing" is happening with consecutive terms. Encourage students to use the word "term," and to be specific when they describe what is happening, for example:
  - "In the first sequence, you always multiply a term by 3 to get the next term."
  - "In the last sequence, if you divide any term by the previous term, you always get $\frac{3}{4}$."
- Tell students that this "same thing" is called the "factor" or "common ratio," and that we will use the second of these. For example, in the second list, the common ratio is 4 because $8 = 4 \cdot 2$, $32 = 4 \cdot 8$, $128 = 4 \cdot 32$, and $512 = 4 \cdot 128$.
- Emphasize that the common ratio is defined to be the multiplier from one term to the next or, said another way, the quotient of a term and the previous term. For example, students may want to say that the pattern of the third sequence is "divide by 2 each time." This is true, but the common ratio is $\frac{1}{2}$, because it is the number you multiply by to get the next term.
- In earlier grades and units, students may have learned that a ratio has two or more parts. As students advance in mathematics, "ratio" is typically used as a synonym for "quotient."

DO THE MATH	PLANNING NOTES

Activity 1: Pieces of Paper (*15 minutes*)

Instructional Routine: Co-Craft Questions (MLR5)	
Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.1a; NC.M1.F-BF.2

In this activity, students generate two geometric sequences from a mathematical situation. The purpose is to create representations of geometric sequences using tables and graphs. At this time, students do not need to write equations for the situation since that work will be the focus of a future lesson.

Note that future lessons will focus on a reasonable domain for sequences when regarded as functions. In this activity, students may "connect the dots" in the graphs with lines, but it's not crucial to address at this time whether or not that's a sensible thing to do.

Step 1

- Students will remain with their same partner.
- Prior to starting the activity, facilitate the *Co-Craft Questions* routine to help students consider the context of this problem.
  - Without revealing the questions that follow, display and read aloud only the task statement that describes Clare's actions: "Clare takes a piece of paper, cuts it in half, then stacks the pieces. She takes the stack of two pieces, then cuts in half again to form four pieces, stacking them. She keeps repeating the process."
  - Ask students to write down possible mathematical questions that could be asked about the situation. Invite three or four students to share their questions with the class. Listen for and amplify any questions that include language related to looking for patterns.

Step 2

- Demonstrate for students the paper cutting described in the task statement, while students complete the first few rows of the table, pausing after each cut.
- After students complete the first few rows of the table ask:
  - "What happens to the number of pieces after each cut?"
  - "What happens to the area of each piece after each cut?"
- Ask students to share their responses with a partner and then invite a few groups to share their response with the class. Ensure students can articulate that as a result of a cut, the number of pieces doubles, and the area of each piece is halved. Students then proceed with the remainder of the activity.

RESPONSIVE STRATEGY

Provide students with the materials (scissors and paper) to cut and stack their paper in place of or along with the demonstration.

Supports accessibility for: Memory; Conceptual processing, Visual-Spatial Processing, and Attention

Monitoring Tip: Monitor for students sketching neat and accurate graphs to highlight during the whole-class discussion and let them know that they may be asked to share later. Include at least one student who does not typically volunteer.

Advancing Student Thinking: If students need support with problem 2, encourage them to think about how the number of pieces in one row of the table relate to the number of pieces in the next and then do the same with the area in successive rows.

Student Task Statement

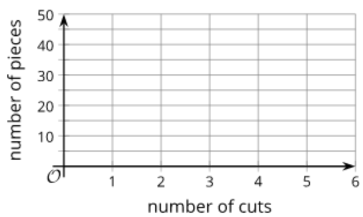
Clare takes a piece of paper, cuts it in half, then stacks the pieces. She takes the stack of two pieces, then cuts in half again to form four pieces, stacking them. She keeps repeating the process.

1. The original piece of paper has length 8 inches and width 10 inches. Complete the table.

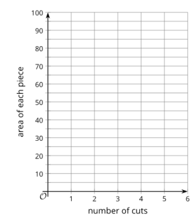
2. Describe in words how you can use the results after 5 cuts to find the results after 6 cuts.

Number of cuts	Number of pieces	Area in square inches of each piece
0		
1		
2		
3		
4		
5		

3. On the given axes, sketch a graph of the number of pieces as a function of the number of cuts. How can you see on the graph how the number of pieces is changing with each cut?



4. On the given axes, sketch a graph of the area of each piece as a function of the number of cuts. How can you see how the area of each piece is changing with each cut?



Are You Ready For More?

1. Clare has a piece of paper that is 8 inches by 10 inches. How many pieces of paper will Clare have if she cuts the paper in half n times? What will the area of each piece be?
2. Why is the product of the number of pieces and the area of each piece always the same? Explain how you know.

Step 3

- The goal of this discussion is for students to identify the common ratio for each sequence and learn that sequences with a common ratio are called geometric sequences.
- Invite previously identified students to share their graphs with the class. Display the two sequences in this activity (1, 2, 4, 8, 16, 32 and 80, 40, 20, 10, 5, 2.5) for all to see.
- Here are some possible questions for discussion:
  - “When did you stop relying on the cutting paper demonstration and complete the table using a pattern?”
  - “What was the pattern you noticed?” (To find the number of pieces, multiply the previous number by 2. To find the area, multiply the previous number by $\frac{1}{2}$.)
  - “How did you find the results after six cuts?” (Multiply 32 by 2 to get 64, and divide 2.5 by 2 to get 1.25.)
  - “What is the factor, or common ratio, for each sequence?” (It’s 2 for the number of pieces and $\frac{1}{2}$ for the area.)
  - “How can you see the common ratio in each graph?” (For the number of pieces, the height of each plotted point is twice the height of the previous plotted point. For the area, the height of each plotted point is half the height of the previous plotted point.)

- Tell students that the sequences they have seen today (in this activity and in the warm-up) have a special name: "geometric sequences." Geometric sequences are characterized by a common ratio. In a geometric sequence, if you divide any term by the previous term, you always get the same value: the common ratio for the sequence. Reiterate that the common ratio for the area sequence is $\frac{1}{2}$ because it's what you multiply by to get the next term.
- Some students may notice the similarity between a geometric sequence and an exponential function. Invite these students to share their observations, such as how both are defined by a factor, referred to as "common ratio" when describing sequences. Tell students that geometric sequences are a type of exponential function and that their knowledge of exponential functions will help them describe geometric sequences during this unit. If students do not bring up the connection to exponential functions, ask, "What do you remember about exponential functions?" Record student responses for all to see and invite comparisons between exponential functions and geometric sequences.

DO THE MATH	PLANNING NOTES

Activity 2: Complete the Sequence (*10 minutes*)

Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.1a; NC.M1.F-BF.2

The purpose of this task is to provide students with practice working with geometric sequences and identifying the common ratio of a sequence. In the lead-up to writing recursive definitions for sequences, it is important for students to understand that for geometric sequences, the common ratio is defined to be the multiplier from one term to the next. Said another way, the common ratio is the quotient of a term and the previous term.

Consider providing access to Desmos, or other assistive technology, so computation does not create a barrier to new learning.

Step 1

- Keep students in their pairs from the last activity.
- Give students quiet think time and then time to share and discuss their answers with their partner.

Advancing Student Thinking: Some students may want to say that the pattern of the third sequence is "divide by 10 each time." Consider asking either question below to help students produce the common ratio:
- "What would you multiply the previous term by, to get the current term?" or
- "What is another way to express dividing by 10?"

Student Task Statement

Complete each geometric sequence in the first column. Then describe the sequence by filling in the blanks in the second column.

1. 1.5, 3, 6, ___, 24, ___	The starting term is ___. The common ratio is ___ because the current term is ____ times the previous term.
2. 40, 120, 360, ___, ___	The starting term is ___. The common ratio is ___ because the current term is ____ times the previous term.
3. 200, 20, 2, ___, 0.02, ___	The starting term is ___. The common ratio is ___ because the current term is ____ times the previous term.
4. $\frac{1}{7}$, ___, $\frac{9}{7}$, $\frac{27}{7}$, ___	The starting term is ___. The common ratio is ___ because the current term is ____ times the previous term.
5. 24, 12, 6, ___, ___	The starting term is ___. The common ratio is ___ because the current term is ____ times the previous term.

Step 2

- For each sequence, invite a student to share how they completed and described the sequence. Highlight the method of dividing any term by the previous term to find the common ratio.
- Emphasize that the presence of a common ratio is what makes a sequence a geometric sequence.

DO THE MATH	PLANNING NOTES

Lesson Debrief (*5 minutes*)

The purpose of this lesson is for students to understand that having a common ratio is what makes a sequence geometric. Students leverage their understanding of exponential functions to support reasoning about geometric sequences and connect the growth/decay factor of an exponential function to the common ratio of a geometric sequence.

	PLANNING NOTES
In the same groups students worked with during the lesson, ask each group to come up with a new geometric sequence and be prepared to explain why it is a geometric sequence. After a brief work time, select three or four groups to share their sequence and explain why it is a geometric sequence. Encourage students to use precise language as they share with the class, such as “term” and “common ratio.” If time allows, ask students if they think the sequence 2, 2, 2, 2, 2 is a geometric sequence. (Yes, it has a common ratio of 1.)	

Student Lesson Summary and Glossary

Consider the sequence 2, 6, 18, . . . How would you describe how to calculate the next term from the previous?

In this case, each term in this sequence is 3 times the term before it.

$$2 \xrightarrow{\cdot 3} 6 \xrightarrow{\cdot 3} 18 \xrightarrow{\cdot 3} \underline{\quad} \xrightarrow{\cdot 3} \underline{\quad}$$

A way to describe this sequence would be: the starting term is 2, and the $current\ term = 3 \cdot previous\ term$.

This is an example of a **geometric sequence**. A geometric sequence is one where the value of each term is the value of the previous term multiplied by a factor. If you know the factor to multiply by, you can use it to find the value of other terms.

> **Geometric sequence**: A sequence in which each term is a constant multiple of the previous term.

This constant multiplier (the “3” in the example) is often called the sequence’s **common ratio**. To find it, you can divide consecutive terms. This can also help you decide whether a sequence is geometric.

> **Common ratio**: The multiplier from one term in a geometric sequence to the next; said another way, the quotient of a term and the previous term

The sequence 1, 3, 5, 7, 9 is not a geometric sequence because $\frac{3}{1} \neq \frac{5}{3} \neq \frac{7}{5}$. The sequence 100, 20, 4, 0.8, however, is geometric because if you divide each term by the previous term you get 0.2 each time: $\frac{20}{100} = \frac{4}{20} = \frac{0.8}{4} = 0.2$.

Cool-down: A Possible Geometric Sequence (*5 minutes*)

Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.1a; NC.M1.F-BF.2
Cool-down Guidance: More Chances Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.	

Cool-down

Here is a sequence: 500, 100, 20, . . .

1. Explain why this sequence could be a geometric sequence.
2. If this sequence is geometric, what is the next term? Explain how you know.

Student Reflection:

Think of today’s class practice. When did you feel most confident? Why is that?

a. Working alone b. Working with one other person c. Whole-class work

DO THE MATH

INDIVIDUAL STUDENT DATA

SUMMARY DATA

NEXT STEPS

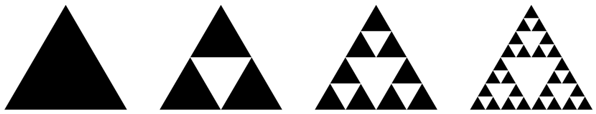
TEACHER REFLECTION

What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?

In the next lesson, students will learn about arithmetic sequences. What do you notice in their work from today's lesson that you might leverage in tomorrow's lesson?

Practice Problems

1. Here are the first two terms of a geometric sequence: 2, 4. What are the next three terms?
2. What is the common ratio of each geometric sequence?
   a. 1,1,1,1,1
   b. 256, 128, 64
   c. 18, 54, 162
   d. 0.8, 0.08, 0.008
   e. 0.008, 0.08, 0.8
3. Compare and contrast the two geometric sequences listed below by considering the starting term and common ratio of each sequence.
   a. Sequence A: 1, 0.5, 0.25, 0.125, 0.0625
   b. Sequence B: 20, 10, 5, 2.5, 1.25
4. A Sierpinski triangle can be created by starting with an equilateral triangle, breaking the triangle into 4 congruent equilateral triangles, and then removing the middle triangle. Starting from a single black equilateral triangle with an area of 256 square inches, here are the first four steps:



Step number	Number of shaded triangles	Area of each shaded triangle in square inches
0	1	256
1	3	
2		
3		
4		
5		

   a. Complete this table showing the number of shaded triangles in each step and the area of each triangle.
   b. Graph the number of shaded triangles as a function of the step number, then separately graph the area of each triangle as a function of the step number.
   c. How are these graphs the same? How are they different?
5. The area of a pond covered by algae is $\frac{1}{4}$ of a square meter on Day 1, and it doubles each day. Complete the table.

Day	1	2	3	4	5	6
Area of algae in square meters						

6. Here is a rule to make a list of numbers:

   Each number is 4 less than 3 times the previous number.

   a. Starting with the number 10, build a sequence of five numbers.

   b. Starting with the number 1, build a sequence of five numbers.

   c. Select a different starting number and build a sequence of five numbers.

   (From Unit 8, Lesson 1)

7. A sequence starts 1, -1, . . .

   a. Give a rule the sequence could follow and the next three terms.

   b. Give a **different** rule the sequence could follow and the next three terms.

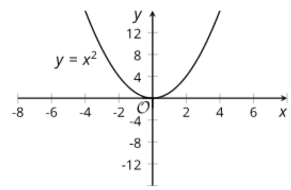
   (From Unit 8, Lesson 1)

8. Which expression in factored form is equivalent to $30x^2 + 31x + 5$?

   a. $(6x+5)(5x+1)$
   b. $(5x+5)(6x+1)$
   c. $(10x+5)(3x+1)$
   d. $(30x+5)(x+1)$

   (From Unit 7)

9. Here is a graph that represents $y = x^2$.



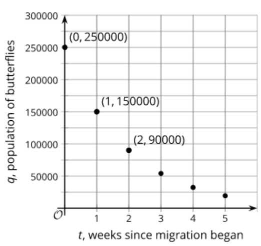
   On the same coordinate plane, sketch and label the graph that represents each equation:

   a. $y = -x^2 - 4$

   b. $y = 2x^2 + 4$

   (From Unit 7)

10. The graph shows a population of butterflies, t weeks since their migration began.



   a. How many butterflies were in the population when they started the migration? Explain how you know.

   b. How many butterflies were in the population after 1 week? What about after 2 weeks?

   c. Write an equation for the population, q, after t weeks.

   (From Unit 6)

11. A person owes $1000 on a credit card that charges an interest rate of 2% per month.

    Complete this table showing the credit card balance each month if they do not make any payments.

Month	Total bill in dollars
1	1,000
2	1,020
3	1,040.40
4	
5	
6	
7	
8	

    (From Unit 6)

12. Which equation has exactly one solution in common with the equation $y = 6x - 2$?

    a. $18x - 3y = 6$
    b. $\frac{1}{2}y = 3x - 2$
    c. $2y = 4x - 12$
    d. $18x - 12 = 3y$

    (From Unit 3)

Lesson 3: Different Types of Sequences

PREPARATION

Lesson Goals	Learning Targets
• Compare and contrast (orally and in writing) arithmetic and geometric sequences. • Determine the rate of change of an arithmetic sequence. • Interpret tables and graphs to determine if a sequence is arithmetic or geometric.	• I can compare and contrast arithmetic and geometric sequences. • I can calculate the rate of change of an arithmetic sequence. • I can tell whether a sequence is arithmetic or geometric by looking at a table or graph.

Lesson Narrative

The purpose of this lesson is for students to understand what makes a sequence an **arithmetic sequence** and to connect it to the idea of a linear function. Arithmetic sequences are characterized by adding a constant value to get from one term to the following term, also known as the **common difference,** just as linear functions are characterized by a constant rate of change.

Building from their thinking about geometric sequences, students begin this lesson by comparing three different sequences. By articulating how the sequences are alike and different, they demonstrate the need for precise language (MP6). Next, students consider two arguments for what type of sequence is represented in a table and then use a graph of the sequence to justify why it could be arithmetic. Throughout the lesson, students will work with and create different representations of functions.

What strategies or representations do you anticipate students might use in this lesson?

Focus and Coherence

Building On	Building Towards
NC.8.F.3: Identify linear functions from tables, equations, and graphs. **NC.M1.F-IF.2:** Use function notation to evaluate linear, quadratic, and exponential functions for inputs in their domains, and interpret statements that use function notation in terms of a context.	**NC.M1.F-IF.3:** Recognize that recursively and explicitly defined sequences are functions whose domain is a subset of the integers, the terms of an arithmetic sequence are a subset of the range of a linear function, and the terms of a geometric sequence are a subset of the range of an exponential function. (*continued*)

NC.M1.F-BF.1a: Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).	**NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). **NC.M1.F-BF.2:** Translate between explicit and recursive forms of arithmetic and geometric sequences and use both to model situations.

Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
  - Technology is required for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U8.L3 Cool-down (print 1 copy per student)

LESSON

Bridge (*Optional, 5 minutes*)

Building On: NC.8.F.3

In this lesson, students will differentiate between arithmetic and geometric sequences. The bridge will emphasize how to determine if functions are linear, which will relate to the arithmetic sequences.

Student Task Statement

One of the tables below represents a linear function, and one does not. Which one represents a linear function? How do you know?

1.

x	0	10	20	30	40	50
y	2.8	10.2	17.6	25	32.4	39.8

2.

x	0	10	20	30	40	50
y	5.6	10.8	16.2	21.8	27.6	33.6

DO THE MATH	PLANNING NOTES

Warm-up: Revisiting Function Notation (*5 minutes*)

Building On: NC.M1.F-IF.2	**Building Towards:** NC.M1.F-IF.3; NC.M1.F-BF.1a

The purpose of this warm-up is to informally assess strategies and understandings students currently have for interpreting function notation which they learned about earlier in the course. Students will use function notation when they define sequences with equations in later lessons, so this warm-up is an opportunity for practice.

Step 1

- Display the first two sentences and ask students to read them quietly. Then, display one problem at a time.
- Give students quiet think time for each problem and ask them to give a signal when they have an answer before displaying the next problem.
- Keep all problems displayed throughout the warm-up.

Advancing Student Thinking: Some students may believe that function notation "distributes"; that is, that expressions like $f(5) - f(4)$ are equivalent to $f(5-4) = f(1)$. Ask these students to work out the value of the expression both ways in order to make clear that $f(5) - f(4)$ and $f(5-4)$ are not equivalent expressions, with a reminder that function notation is not multiplication but rather a way to write the output of a function for a specific input.

Student Task Statement

Consider the function f given by $f(n) = 3n - 7$. This function takes an input, multiplies it by 3, then subtracts 7.

Evaluate mentally.

1. $f(10)$
2. $f(10) - 1$
3. $f(10-1)$
4. $f(5) - f(4)$

Step 2

- Ask students to share their strategies for each problem.
- Record and display their responses for all to see.

DO THE MATH	PLANNING NOTES

Activity 1: Three Sequences (*15 minutes*)

Instructional Routine: Collect and Display (MLR2)	
Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-IF.3; NC.M1.F-BF.2

The purpose of this activity is for students to contrast three different types of sequences and to introduce the term "arithmetic sequence."

Step 1

- Provide access to Desmos or other assistive technology.
- Ask students to arrange themselves in pairs or use visibly random grouping.
- Give students quiet work time and then time to share their work with a partner.
- As students discuss the sequences, use the *Collect and Display* routine.
  - Circulate and listen to students talk about the patterns they notice.
  - Write down common or useful phrases you hear students say about each type onto a visual display, for example "add 10 each time" or "multiply each term by 2."
  - Collect the responses into a visual display.
  - Throughout the remainder of the lesson, continue to update collected student language and remind students to borrow language from the display as needed.

Monitoring Tip: Monitor for students using precise language, either orally or in writing, during work time to invite to share during the whole-class discussion.

***Advancing Student Thinking*:** Students may need help identifying a pattern for the two non-geometric sequences. Ask, "What can you say about the change between consecutive terms in the sequence?" Encourage students to use any language they wish to describe the pattern—they do not need to use an equation at this time. Some students may benefit from creating a table where they can see the term number and the value of the term side by side, particularly for sequence B.

Student Task Statement

Here are the values of the first five terms of three sequences:

A: 30, 40, 50, 60, 70, . . . B: 0, 5, 15, 30, 50, . . . C: 1, 2, 4, 8, 16, . . .

1. For each sequence, describe a way to produce a new term from the previous term.
2. If the patterns you described continue, which sequence has the smallest value for the 10th term?
3. Which of these could be geometric sequences? Explain how you know.

Are You Ready For More?

Elena says that it's not possible to have a sequence of numbers that is *both* arithmetic and geometric. Do you agree with Elena? Explain your reasoning.

Step 2

- Facilitate a whole-class discussion. The purpose of this discussion is to compare different types of sequences and introduce students to the term "arithmetic sequence." Begin the discussion by asking students how A and C are alike and different. They might offer things like:
  - C is geometric, but A is not.
  - In A, you always add 10 to get from term to term, but in C, you always multiply by 2.
  - In C, the common ratio is 2. In A, you get the next term by adding 10 to the previous term.

  During this discussion, remind students to borrow and build on any relevant language that was collected and displayed. Add to the display if new words and phrases are used to clarify the displayed language.
- Tell students that sequence A is an example of an arithmetic sequence. Scribe "arithmetic sequence" on the display, and then annotate the display to indicate any student language that is about the arithmetic sequence.
- Here are two ways to know a sequence is arithmetic:
  - You always add the same number to get from one term to the next.
  - If you subtract any term from the next term, you always get the same number.
- Share that the number added to get the next term in an arithmetic sequence is called the "common difference." In sequence A, the common difference is 10, because $40 = 30 + 10$, $50 = 40 + 10$, $60 = 50 + 10$, and $70 = 60 + 10$.
- Also share with students that you can describe an arithmetic sequence best by providing the starting term and the common difference. For example, sequence A could be described as "the starting term is 30 and the current term = previous term + 10."
- Ask students how they would classify B and why. (B does not have a common ratio or common difference.)
- Some students may notice the similarity between an arithmetic sequence and a linear function. Invite these students to share their observations.
  - They may relate the constant rate of change of a linear function to the common difference of an arithmetic sequence. Tell students that arithmetic sequences are a type of linear function and that their knowledge of linear functions will help them describe arithmetic sequences during this unit. If students do not bring up the connection to linear functions, ask, "What do you remember about linear functions?" Record student responses and invite comparisons between linear functions and arithmetic sequences.

RESPONSIVE STRATEGY

Ask students to sort the collected language into three groups, one for language used to describe arithmetic sequences, a second for geometric sequences, and the third for neither.

Supports accessibility for: Language; Memory; Conceptual processing

DO THE MATH	PLANNING NOTES

Activity 2: Representing a Sequence (*10 minutes*)

Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.1a

The purpose of this activity is for students to create another representation of a given sequence and to give them an opportunity to use the vocabulary they have learned for geometric and arithmetic sequences.

Step 1

- Give students some time to work on the task independently.

> **Monitoring Tip:** Monitor for students who create Mai's graph in order to understand her reasoning and for students who reason about whether the sequence is defined by a common difference or a common ratio. Let them know that they may be asked to share later.

Student Task Statement

Jada and Mai are trying to decide what type of sequence this could be:

Term number	Value
1	2
2	6
5	18

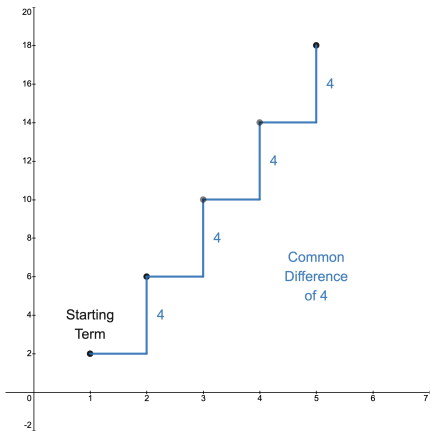
Jada says: "I think this sequence is geometric because in the value column each row is 3 times the previous row."

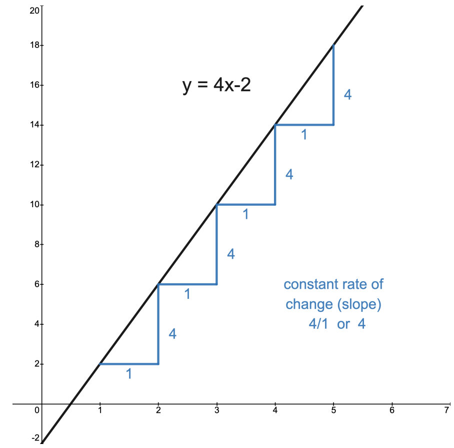
Mai says: "I don't think it is geometric. I graphed it, and it doesn't look geometric."

Do you agree with Jada or Mai? Explain or show your reasoning.

Step 2

- Invite previously selected students to share their reasoning, displaying any graphs created for all to see. If the idea of slope or constant rate of change does not come up, ask students how they could have found the missing points for an arithmetic sequence just given $(2, 6)$ and $(5, 18)$. Draw in the slope triangle and show the computation for the constant rate of change. If not already discussed, ask students why they think Jada may have thought the sequence was geometric. (Jada might not have noticed that the term numbers weren't all going up by one.)
- Explicitly connect the idea of constant rate of change of linear functions to the new vocabulary of "common difference" used to describe arithmetic sequences. Display a graph of a linear function, alongside a graph of the corresponding arithmetic sequence. Reference the diagrams as you explain, "In a linear function, we can think of the constant rate of change as the amount the output increases when the input increases by 1. In an arithmetic sequence, the input is the term number, and the output is the value of the term. As the term number increases by 1, the value of the term changes by the common difference."





DO THE MATH	PLANNING NOTES

Lesson Debrief (*5 minutes*)

The purpose of this lesson is for students to understand what makes a sequence an arithmetic sequence and to connect this to the idea of a linear function.

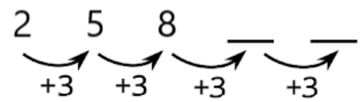
	PLANNING NOTES

Ask students to discuss with a partner: "How are arithmetic and geometric sequences alike and different?" After they have had a few minutes to discuss, ask several students to share the things that came up. Some things to highlight:
• They are both types of sequences. That is, they are both lists of numbers.
• To get from one term to the next for both arithmetic and geometric sequences, you "do the same thing each time."
  – For geometric sequences, you always multiply by the same number to get the next term—the common ratio.
  – For arithmetic sequences, you always add the same number to get the next term—the common difference.
• Geometric and arithmetic sequences are both connected to functions we have learned about before.
  – Geometric sequences have common ratios (the quotient of any term and the previous term), which will be equivalent to the growth or decay factors of the related exponential function.
  – Arithmetic sequences have a common difference (the difference of any term and the previous term), which will be equivalent to the constant rate of change of the related linear function.	

Student Lesson Summary and Glossary

Consider the sequence 2, 5, 8, . . . How would you describe how to calculate the next term from the previous?

In this case, each term in this sequence is 3 more than the term before it.



A way to describe this sequence is: the starting term is 2 and the $\textit{current term} = \textit{previous term} + 3$.

This is an example of an **arithmetic sequence**. An arithmetic sequence is one where each term is the sum of a previous term and a constant. (In the previous example, the constant is 3.) If you know the constant to add, you can use it to find other terms.

> **Arithmetic sequence:** A sequence in which each term comes from adding a constant to the previous term.

For example, each term in this sequence is 3 more than the term before it. To find this constant, called the **common difference**, you can subtract consecutive terms. This can also help you decide whether a sequence is arithmetic.

> **Common difference**: The number to add to get from one term in an arithmetic sequence to the next; said another way, the difference of a term and the previous term.

For example, the sequence 3, 6, 12, 24 is not an arithmetic sequence because $6-3 \neq 12-6 \neq 24-12$. But the sequence 100, 80, 60, 40 is because if the differences of consecutive terms are all the same: $80-100 = 60-80 = 40-60 = -20$. This means that the rate of change is -20 for the sequence 100, 80, 60, 40.

It is important to remember that while the last two lessons have introduced geometric and arithmetic sequences, there are many other sequences that are neither geometric nor arithmetic.

Cool-down: Do What's Next (*5 minutes*)

Building Towards: NC.M1.F-BF.1a; NC.M1.F-BF.2

Cool-down Guidance: Points to Emphasize
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding. Both Lessons 4 and 5 include arithmetic and geometric sequences for students to practice identifying and generating, but the focus is on new aspects (spreadsheets and functions). If students struggle with this cool-down, emphasize in these lessons how to determine if a sequence is arithmetic or geometric and how to use that information to find terms in the sequence.

Cool-down

Many sequences start with the terms 2 and 8.

1. Find the next two terms of the arithmetic sequence 2, 8, ____, ____.
2. Find the next two terms of the geometric sequence 2, 8, ____, ____.
3. Find two possible next terms of a sequence 2, 8, ____, ____ that is neither geometric nor arithmetic.

Student Reflection:

A classmate needs help understanding the difference between arithmetic and geometric sequences. How would you best describe this to them?

DO THE MATH

INDIVIDUAL STUDENT DATA

SUMMARY DATA

NEXT STEPS

TEACHER REFLECTION

What part of the lesson went really well today in terms of students' learning? What did you do that made that part go well? What happened today that will influence the planning of future lessons?

Think about a time you recently made a mistake during math class. How did you leverage your mistake to show students that mistakes are just learning in process?

Practice Problems

1. The first two terms of some different arithmetic sequences are listed below. What are the next three terms of each sequence?

   a. -2, 4

   b. 11, 111

   c. 5, 7.5

   d. 5, -4

2. For each sequence, decide whether it could be arithmetic, geometric, or neither.

    a. 200, 40, 8, . . .

    b. 2, 4, 16, . . .

    c. 10, 20, 30, . . .

    d. 100, 20, 4, . . .

    e. 6, 12, 18, . . .

3. Complete each arithmetic sequence with its missing terms, then state the common difference for each sequence.

    a. -3, -2, ___, ___, 1

    b. ___, 13, 25, ___, ___

    c. 1, .25, ___, -1.25, ___

    d. 92, ___, ___ ,___, 80

4. A sequence starts with the terms 1 and 10.

    a. Find the next two terms if it is arithmetic: 1, 10, ___, ___.

    b. Find the next two terms if it is geometric: 1, 10, ___, ___.

    c. Find two possible next terms if it is neither arithmetic nor geometric: 1, 10, ___, ___.

5. For each sequence, decide whether it could be arithmetic, geometric, or neither.

    a. 25, 5, 1, …

    b. 25, 19, 13, …

    c. 4, 9, 16, …

    d. 50, 60, 70, …

    e. $\frac{1}{2}$, 3, 18, …

6. Complete each geometric sequence with the missing terms. Then find the common ratio for each.

    a. ___, 5, 25, ___, 625

    b. -1, ___, -36, 216, ___

    c. 10, 5, ___, ___, 0.625

    d. ___, ___, 36, -108, ___

    e. ___, 12, 18, 27, ___

    (From Unit 8, Lesson 2)

7. The first term of a sequence is 4.

    a. Choose a common ratio and list the next three terms of a geometric sequence.

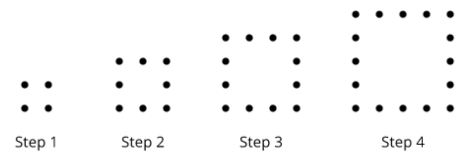
    b. Choose a **different** common ratio and list the next three terms of a geometric sequence.

    (From Unit 8, Lesson 2)

8. Han says this pattern of dots can be represented by a quadratic relationship because the dots are arranged in a rectangle in each step.

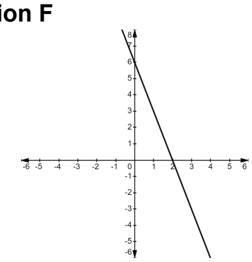
   Do you agree? Explain your reasoning.

   (From Unit 7)



9. Review the functions below.

Function F



Function G

$g(x) = \frac{1}{2}(x+2)(x-4)$

   a. Over what domain are both functions decreasing?

   b. Which function has the larger y-intercept?

   (From Unit 7)

10. Match each quadratic expression given in factored form with an equivalent expression in standard form. One expression in standard form has no match.

   a. $(2+x)(2-x)$

   b. $(x+9)(x-9)$

   c. $(2+x)(x-2)$

   d. $(x+y)(x-y)$

   1. x^2-4

   2. $81-x^2$

   3. x^2-y^2

   4. $4-x^2$

   5. x^2-81

   (From Unit 7)

11. Solve each system of equations.

   a. $\begin{cases} 7x-12y=180 \\ 7x=84 \end{cases}$

   b. $\begin{cases} -16y=4x \\ 4x+27y=11 \end{cases}$

   (From Unit 3)

12. Does the following table represent a linear or non-linear function? Explain your answer.

x	0	1	3	6	8
y	5	10	15	20	25

(Addressing NC.8.F.3)

Lesson 4: Sequences Are Functions

PREPARATION

Lesson Goals	Learning Targets
• Comprehend that sequences are functions whose domain is a subset of the integers. • Create (in writing) a recursive definition for a sequence using function notation.	• I can describe a sequence as a function and give its domain. • I can define arithmetic and geometric sequences recursively using function notation.

Lesson Narrative

Building on the informal language students have used so far in the unit, the purpose of this lesson is for students to understand that sequences are functions and to use function notation when defining them with equations. In previous lessons, they described the arithmetic sequence 99, 96, 93, . . . as starting at 99 where each term is 3 less than the previous term. Now they think of this sequence as a function f, and write $f(1) = 99$ and $f(n) = f(n-1) - 3$ for $n \geq 2$, where n is an integer. This is called a **recursive definition** for f because it describes a repeated, or recurring, process for getting the values of f, namely the process of subtracting 3 each time. Students will use recursive definitions to describe functions in both mathematical and real-world contexts throughout the remainder of this unit.

In the warm-up, students make sense of a dot pattern as a function where the number of dots in each step depends on the step number (MP1). This helps prepare students to write a recursive definition for the function by expressing regularity in repeated reasoning while using a table in the following activity (MP8). Also during this activity, students decide what values make sense for the domain of the function, which leads to expanding their definition of sequence to a function whose domain is a subset of the integers. Students then return to sequences they have seen previously in the unit and define them recursively using function notation.

Regarding the notation students use when writing an equation that defines a sequence, it is more important that students can encapsulate the rule correctly than it is that they can do it in a particular format. Throughout this unit, definitions of sequences written in function notation are always followed by an inequality using n that describes the domain of the function, which is particularly helpful when students work with definitions for the n^{th} term of a sequence and need to indicate what the first term of the sequence is. This is not meant to imply that all students should use such inequalities every time they define a sequence using function notation. It may be better:

- For students to write out in words, instead of inequalities, the restrictions on the domain of a sequence.
- To keep the focus on writing equations and tell students to be prepared to respond orally when asked about the domain of a sequence.

What are you excited for your students to be able to do after this lesson?

Focus and Coherence

Building On	Addressing	Building Towards
NC.M1.F-IF.1: Build an understanding that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range by recognizing that: • if f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. • the graph of f is the graph of the equation $y = f(x)$. **NC.M1.F-IF.2:** Use function notation to evaluate linear, quadratic, and exponential functions for inputs in their domains, and interpret statements that use function notation in terms of a context.	**NC.M1.F-IF.3:** Recognize that recursively and explicitly defined sequences are functions whose domain is a subset of the integers, the terms of an arithmetic sequence are a subset of the range of a linear function, and the terms of a geometric sequence are a subset of the range of an exponential function. **NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).	**NC.M1.F-BF.2:** Translate between explicit and recursive forms of arithmetic and geometric sequences and use both to model situations.

Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U8.L4 Cool-down (print 1 copy per student)

LESSON

Bridge (*Optional, 5 minutes*)

Building On: NC.M1.F-IF.1

In this bridge, students encounter one function with a discrete domain and another with a continuous domain. The purpose is to prime students to recognize the domain of sequences as a subset of the integers.

Student Task Statement

What are the domains of the functions below? Explain your answers.

1. Students' scores on Ms. Starks' Math 1 test are modeled by a function that outputs a score based on the number of questions answered correctly. There are 12 questions on the test.

2. A medical research facility conducted a study using a function to determine healthy body weight based on patients' heights. The patients' heights in the study varied between 58 inches and 79 inches.

DO THE MATH	PLANNING NOTES

Warm-up: Bowling for Triangles (Part One) (*5 minutes*)

Building Towards: NC.M1.F-IF.3

Up until this lesson, students have described sequences using mostly informal language such as:

- Geometric: “The starting term is ___, and the current term = ___ · previous term.”
- Arithmetic: “The starting term is ___, and the current term = previous term + ___.”

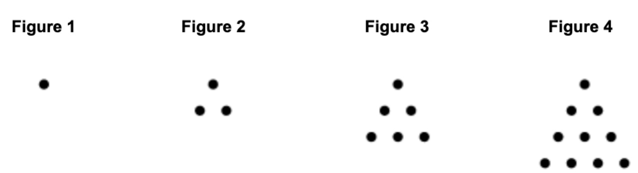
The goal of this activity is for students to use that type of language to describe a pattern of dots and make sense of the general structure. In the following activity, students will continue working with the same pattern as they see how we can use function notation to recursively define sequences (MP1).

Step 1

- Ask students to close their books or devices for the warm-up.
- Display the image for all to see.

Student Task Statement

Use the sentence frame to describe how to produce one figure of the pattern from the previous figure by completing the provided sentence.



The starting term is _____, and the current term = ____ .

Step 2

- Select students to share their descriptions, recording these for all to see next to the image.
- If none of the descriptions includes mention of the figure number, ask, “Does anyone see how the change between figures is related to the number of the figure in the sequence?”
- Next, say, “The number of dots in a figure is a function of the figure number. Let's call this function D. What is the value of $D(4)$ and $D(5)$?” (10 and 15.) If needed, point out that the input of this function is the figure number, and the output is the number of dots in the patterns.
- After some quiet think time, select students to share their values and explain their reasoning.

DO THE MATH	PLANNING NOTES

Activity 1: Bowling for Triangles (Part Two) (*15 minutes*)

Building On: NC.M1.F-IF.2	**Addressing:** NC.M1.F-BF.1a; NC.M1.F-IF.3

Continuing with the dot pattern from the warm-up, the goal of this activity is for students to understand that since sequences are functions, we can take the recursive definition they stated in the warm-up and express it using function notation. Students use a table to express regularity in repeated reasoning as they write an expression for $D(n)$ using $D(n-1)$ (MP8). They also consider possible inputs to D, reasoning that only integer values make sense. This leads to expanding their understanding of sequences as functions whose inputs are restricted to the integers.

The sequence presented here is not arithmetic or geometric. The use of a sequence that is not arithmetic or geometric helps distinguish the explicit exponential and linear function rules, which students are very familiar with, from this newly defined recursive rule.

Step 1

- Ask students to arrange themselves in pairs or use visibly random grouping. Students will remain with these partners for the remainder of the lesson.
- Provide time for students to work independently on questions 1 and 2. After students have completed the table in question 2, encourage partners to compare table entries and resolve any differences.
- Once partners agree on question 2, ask partners to collaboratively arrive at an equation for question 3.

RESPONSIVE STRATEGY

Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem.

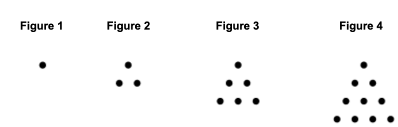
Supports accessibility for: Visual-spatial processing

Advancing Student Thinking: If students are unfamiliar with the term "integer," remind them that the integers are all the counting numbers and their opposites, including zero. One way to write the integers is . . . ,-3, -2, -1, 0, 1, 2, 3, . . .

If students are unclear on how to approach question 3, point out that the previous figure's term number is $(n-1)$. If they continue to struggle, share that the number of dots in the previous figure can be represented by $D(n-1)$.

Student Task Statement

Here is a visual pattern of dots. The number of dots $D(n)$ is a function of the figure number n.



1. What values make sense for n in this situation? Would $n = 2.5$ make sense?
2. Complete the table for Figures 1 to 5.
3. Following the pattern in the table, write an equation for $D(n)$ in terms of the previous figure. Be prepared to explain your reasoning.

n	$D(n)$
1	1
2	$D(1) + 2 = 3$
3	$D(2) + 3 = 6$
4	
5	

Are You Ready For More?

Consider the same triangular pattern.

1. Is the sequence defined by the number of dots in each step arithmetic, geometric, or neither? Explain how you know.
2. Can you write an expression for the number of dots in Step n without using the value of D from a previous step?

Step 2

- The two important takeaways from this discussion are that sequences are a type of function whose domain is a subset of the integers and an understanding of what $D(n)$ and $D(n-1)$ mean.
- Begin the discussion by inviting students to share values that do and do not make sense for n, recording these for all to see. Make sure students understand that non-integer values do not make sense for the sequence represented by this dot pattern since there is no partial figure between the figures that we can calculate. If students ask about $D(0)$, let them know that we could say for the pattern that there is a Figure 0 with 0 dots. $D(-1)$, however, does not make sense here unless we attempt to define what a "negative dot" is. Tell students that an important aspect of working with functions is defining a domain that makes sense given the context and what they are trying to do.
- Next, invite students to share their equation for $D(n)$, focusing on how they reasoned about "in terms of the previous figure" in order to get to $D(n) = D(n-1) + n$. Tell students that this is known as a "recursive definition" because it describes a repeated, or recurring, process for getting the values of D. The two other pieces needed for this type of definition are what value to start with and what values n can be. In this case, we can write the full definition as $D(1) = 1$ and $D(n) = D(n-1) + n$ for $n \geq 2$, where n is an integer. Often times the "n is an integer" piece is left off when we know the function is a sequence since the domain of a sequence is always a subset of the integers.
- Using $n-1$ as the input to a function is likely an unfamiliar idea for students. They will continue to practice using function notation to define sequences recursively in the following activity, so they do not need to have mastery at this time.

DO THE MATH	PLANNING NOTES

Activity 2: Let's Define Some Sequences (*10 minutes*)

Instructional Routine: Discussion Supports (MLR8)	
Addressing: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.2

In the previous activity, students began the work of writing recursive definitions for sequences using function notation; the purpose of this activity is for students to get more practice writing such definitions. Several of the sequences selected are ones students have seen previously since the focus of this activity is on the notation and not on making sense of the sequence. The sequence 1, 3, 7, 15, 31, . . . is from the Tower of Hanoi puzzle students engaged with in Unit 7, Lesson 27 and is the only sequence in the activity that is neither arithmetic nor geometric.

Step 1

- After quiet work time, ask students to compare their definitions for sequences C and D to their partner's and decide if they are both correct, even if they are different. Follow with a whole-class discussion.

Monitoring Tip: Monitor for groups who wrote their definitions for the first and third sequences (C and D) differently to share during the discussion and let them know that they may be asked to share later. Include at least one student who does not typically volunteer. For example, for the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},$
$B(n) = B(n-1) \cdot 0.5$, $B(n) = \frac{B(n-1)}{2}$, and $B(n) = \frac{1}{2} \cdot B(n-1)$ ($B(1) = 1$ with $n \geq 2$) are three possible definitions.

Advancing Student Thinking: Some students may not be sure where to start when defining a function recursively. Encourage them to look back at the previous activity. A good intermediate step is to use the following formats students saw previously:

- Geometric: the starting term is ___, and the current term = ___ · previous term
- Arithmetic: the starting term is ___, and the current term = previous term + ___.

Student Task Statement

Use the first five terms of each sequence to state if the sequence is arithmetic, geometric, or neither. Next, define the sequence recursively using function notation.

1. A: 30, 40, 50, 60, 70, . . .
2. B: 1, 2, 4, 8, 16, 32, . . .
3. C: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
4. D: 20, 13, 6, -1, -8, . . .
5. E: 1, 3, 7, 15, 31, . . .

Step 2

- Facilitate a discussion for students to notice some common features of arithmetic and geometric sequences.
    - Display the five sequences for all to see throughout the discussion to record the sequence type and student thinking next to each sequence.
    - For the sequences C and D, invite previously identified groups to first share the sequence type and then the different ways they wrote their definitions. Repeat this process for the remaining sequences until the sequence type and at least one definition are written for each sequence.
    - Use *Discussion Supports* to support whole-class discussion. After each student shares, provide the class with the following sentence frames to help them respond: "I agree because..." or "I disagree because...." If necessary, revoice student ideas to demonstrate mathematical language use by restating a statement as a question in order to clarify, apply appropriate language, and involve more students.
- Consider calling out that $A(n)$ and $A(n-1)$ represent the outputs of the function A for an input of n and $n-1$. These should be treated just like a number and cannot be rewritten as $An - A$ using the distributive property or $(n-1)A$ using the commutative property.
- Conclude the discussion by inviting students to share things they notice about the definitions for the different sequence types. (The arithmetic sequences have adding or subtracting a quantity; the geometric sequences have multiplying by a factor, and the sequence that is neither is a mix of adding and multiplying.) If not brought up by students, remind them that arithmetic sequences are a type of linear function, and geometric sequences are a type of exponential function, so it makes sense to see only addition and subtraction for arithmetic sequences and only multiplication for geometric sequences.

DO THE MATH

PLANNING NOTES

Lesson Debrief (*5 minutes*)

The purpose of this lesson is to introduce students to the idea that sequences are a type of function and to use function notation to describe sequences they had previously described only informally.

As a whole class, make one display for geometric and arithmetic sequences. Ask students to participate in filling in the table. For example, in the second row, leave blank spaces before the words "exponential" and "linear". Ask students to work with a partner to discuss what missing terms are needed and then discuss as a whole class.

This display should be posted in the classroom for the remaining lessons within this unit. It should look something like what is shown here. In the next lesson, you will add equations using subscript notation to the display. Leave enough room to add more examples from future lessons.

Sequence: A List of Numbers	
Geometric (exponential function)	**Arithmetic** (linear function)
Multiply each term by the factor (common ratio).	Add to each term the constant rate of change (common difference).
Example 1 Sequence: 2, 6, 18, 54, . . . Common Ratio (growth factor): 3 Recursive: $f(1) = 2, f(n) = 3 \cdot f(n-1)$, $n \geq 2$	**Example 1** Sequence: 2, 7, 12, 17, . . . Common Difference (constant rate of change): 5 Recursive: $g(1) = 2,\ g(n) = 5 + g(n-1)$, $n \geq 2$
Example 2 Sequence: 160, 40, 10, 2.5, . . . Common Ratio (decay factor): $\frac{1}{4}$ Recursive: $h(1) = 160,\ h(n) = \frac{1}{4} \cdot h(n-1)$, $n \geq 2$	**Example 2** Sequence: 9, 5, 1, -3, . . . Common Difference (constant rate of change): -4 Recursive: $k(1) = 9,\ k(n) = -4 + k(n-1)$, $n \geq 2$

PLANNING NOTES

Student Lesson Summary and Glossary

Sometimes we can define a sequence recursively. That is, we can describe how to calculate the next term in a sequence if we know the previous term.

Here's a sequence: 6, 10, 14, 18, 22, . . . This is an arithmetic sequence, where each term is 4 more than the previous term. Since sequences are functions, let's call this sequence f .Then we can use function notation to write $f(n) = f(n-1) + 4$. Here, $f(n)$ is the term, $f(n-1)$ is the previous term, and + 4 represents the common difference.

When we define a function recursively, we also must say what the first term is. Without that, there would be no way of knowing if the sequence defined by $f(n) = f(n-1) + 4$ started with 6 or 81 or any other number. Here, one possible starting value is $f(1) = 6$. (It could also make sense to number the terms starting with 0, using $f(0) = 6$, and we'll talk more about this later.)

Combining this information gives the recursive definition: $f(1) = 6$ and $f(n) = f(n-1) + 4$ for $n \geq 2$, where n is an integer. We include the $n \geq 2$ at the end since the value of f at 1 is already given and the other terms in the sequence are generated by inputting integers larger than 1 into the definition.

Recursive definition (of a sequence): A way of writing a rule for the terms of sequence that depends on the terms that came before. For example, the sequence a: 10, 8, 6, 4… has recursive definition $a(1) = 10$ and $a(n) = a(n-1) - 2$ for $n \geq 2$.

Cool-down: Define This Sequence (*5 minutes*)

Addressing: NC.M1.F-BF.1a

Cool-down Guidance: Points to Emphasize
If students struggle on this cool-down, emphasize student responses to the discussion in Lesson 5's warm-up, paying close attention to how to determine whether a sequence is arithmetic or geometric and how to calculate the common difference or ratio. Use the recursive presentation in question 4 to show how the first five terms of the sequence connect to the equation in function notation. Monitor student progress in Activity 1 of Lesson 5.

Cool-down

Use the first five terms of sequence H to define the sequence recursively using function notation.

2.5, 7.5, 22.5, 67.5, 202.5, . . .

Student Reflection:

What connections do you see between the concepts you've learned in this unit so far to the world around you?

DO THE MATH

INDIVIDUAL STUDENT DATA

SUMMARY DATA

NEXT STEPS

TEACHER REFLECTION

What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?

Why is it important for students to be able to write and use different forms (verbal descriptions and function notation) to define arithmetic and geometric sequences?

Practice Problems

1. Match each sequence with one of the definitions. Note that only the part of the definition showing the relationship between the current term and the previous term is given so as not to give away the solutions.
   a. 6, 12, 18, 24
   b. 2, 14, 98, 686
   c. 160, 80, 40, 20

   1. $a(n) = 7 \cdot a(n-1)$
   2. $b(n) = \frac{1}{2} \cdot b(n-1)$
   3. $c(n) = c(n-1) + 6$

2. Write the first five terms of each sequence. Determine whether each sequence is arithmetic, geometric, or neither.
   a. $a(1) = 7, a(n) = a(n-1) - 3$ for $n \geq 2$.
   b. $b(1) = 2, b(n) = 2 \cdot b(n-1) - 1$ for $n \geq 2$.
   c. $c(1) = 3, c(n) = 10 \cdot c(n-1)$ for $n \geq 2$.
   d. $d(1) = 1, d(n) = n \cdot d(n-1)$ for $n \geq 2$.

3. Match each recursive definition with one of the sequences.
   a. $h(1) = 1, h(n) = 2 \cdot h(n-1) + 1$ for $n \geq 2$
   b. $p(1) = 1, p(n) = 2 \cdot p(n-1)$ for $n \geq 2$
   c. $a(1) = 80, a(n) = \frac{1}{2} \cdot a(n-1)$ for $n \geq 2$

   1. 80, 40, 20, 10, 5
   2. 1, 2, 4, 8, 16
   3. 1, 3, 7, 15, 31

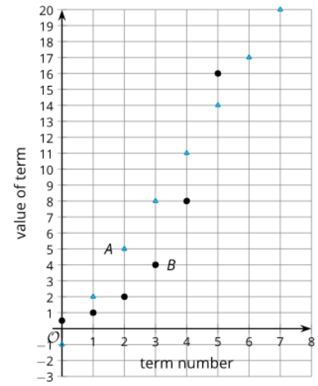
4. Here is the graph of two sequences. Complete the table for each sequence.

    a. For sequence A, describe a way to produce a new term from the previous term.

    b. For sequence B, describe a way to produce a new term from the previous term.

    c. Which of these is a geometric sequence? Explain how you know.

Term number	Sequence A	Sequence B
0	-1	$\frac{1}{2}$
1		
2		
3		
4		
5		
6		



(From Unit 8, Lesson 3)

5. The first five terms of some sequences are given. State a rule that each sequence could follow.

    a. 2, 4, 6, 8, 10

    b. 5, 7, 9, 11, 13

    c. 50, 25, 0, -25, -50

    d. $\frac{1}{3}$, 1, 3, 9, 27

(From Unit 8, Lesson 1)

6.

    a. Describe the graph of $y = -x^2$. (Does it open upward or downward? Where is its y-intercept? What about its x-intercepts?)

    b. Without graphing, describe how adding $16x$ to $-x^2$ would change each feature of the graph of $y = -x^2$. (If you get stuck, consider writing the expression in factored form.)

        i. the x-intercepts

        ii. the vertex

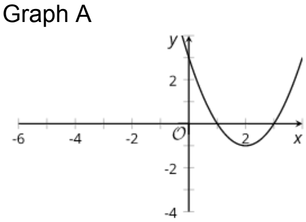
        iii. the y-intercept

        iv. the direction of opening of the U-shape graph

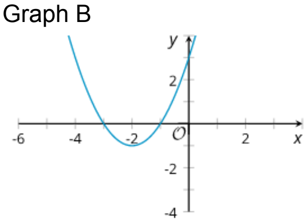
(From Unit 7)

7. Here are four graphs. Match each graph with a quadratic equation that it represents.

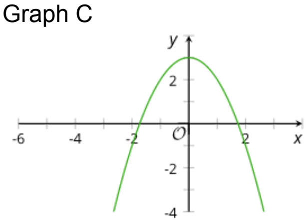
Graph A



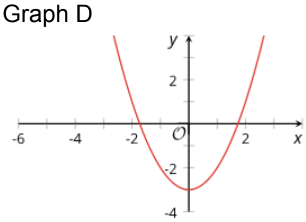
Graph B



Graph C



Graph D



a. $y = -x^2 + 3$
b. $y = (x+1)(x+3)$
c. $y = x^2 - 3$
d. $y = (x-1)(x-3)$

(From Unit 7)

8. What are the x-intercepts of the graph of the function defined by $(x-2)(2x+1)$?

   a. $(2,0)$, and $(-1,0)$

   b. $(2,0)$ and $(-\frac{1}{2},0)$

   c. $(-2,0)$ and $(1,0)$

   d. $(-2,0)$ and $(\frac{1}{2},0)$

   (From Unit 7)

9. (*Technology required*). Function h is defined by $h(x)=5x+7$, and function k is defined by $k(x)=(1.005)^x$.

   a. Complete the table with values of $h(x)$ and $k(x)$. When necessary, round to 2 decimal places.

x	$h(x)$	$k(x)$
1		
10		
50		
100		

   b. Which function do you think *eventually* grows faster? Explain your reasoning.

   c. Use graphing technology to verify your answer to the previous question.

   (From Unit 6)

10. At 6:00 a.m., Lin began hiking. At noon, she had hiked 12 miles. At 4:00 p.m., Lin finished hiking with a total trip of 26 miles. During which time interval was Lin hiking faster? Explain how you know.

    (From Unit 5)

11. Function f is defined by $f(x)=2x-7$, and g is defined by $g(x)=5^x$.

    a. Find $f(3)$, $f(2)$, $f(1)$, $f(0)$, and $f(-1)$.

    b. Find $g(3)$, $g(2)$, $g(1)$, $g(0)$, and $g(-1)$.

    (From Unit 5)

Lesson 5: Representing Sequences

PREPARATION

Lesson Goals	Learning Targets
• Interpret a recursive definition for a sequence written using function or subscript notation. • Create (in writing) a recursive definition for a sequence using subscript notation.	• I can interpret sequences that are expressed in function or subscript notation. • I can define arithmetic and geometric sequences using subscript notation.

Lesson Narrative

In the previous lesson, students used function notation to define a sequence with an equation. Subscript notation is also used when writing these definitions. The goal of the lesson is for students to practice interpreting and writing recursive definitions of functions using both function notation and subscript notation.

In the first activity, students match sequences and recursive definitions, giving them opportunities to explain their reasoning and critique the reasoning of others (MP3). In the second activity, students look for and make use of structure (MP7) to interpret subscript notation for a recursive definition. An important part of the activities is giving students time to share and explain their strategies for interpreting and writing these definitions.

Which Standards for Mathematical Practice do you anticipate students engaging in during this lesson? How will you support them?

Focus and Coherence

Building On	Addressing	Building Towards
NC.6.EE.2: Write, read, and evaluate algebraic expressions. • Write expressions that record operations with numbers and with letters standing for numbers. • Identify parts of an expression using mathematical terms and view one or more of those parts as a single entity. • Evaluate expressions at specific values of their variables using expressions that arise from formulas used in real-world problems.	**NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities. Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).	**NC.M1.F-BF.2:** Translate between explicit and recursive forms of arithmetic and geometric sequences and use both to model situations.

Agenda, Materials, and Preparation

Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate technology to solve problems. It is ideal if each student has accessibility to their own device.

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U8.L5 Cool-down (print 1 copy per student)

LESSON

Bridge (*Optional, 5 minutes*)

Building On: NC.6.EE.2

The purpose of this bridge is to examine values of the expressions $n-1$, n, and $n+1$. These expressions are used when defining sequences using function or subscript notation to indicate the term number. In this task, students evaluate the algebraic expressions and discuss the pattern.

Student Task Statement

1. Evaluate each expression given $n=4$.
   a. $n-1$
   b. n
   c. $n+1$
2. What do you notice about the values of the three expressions? Explain why this is true.

DO THE MATH	PLANNING NOTES

Warm-up: Reading Representations (*5 minutes*)

Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.1a

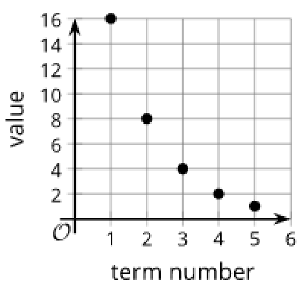
The purpose of this warm-up is for students to recall some of the ways functions can be represented, such as tables, graphs, equations, and descriptions. The focus here is on exponential and linear functions, and identifying either the common ratio or common difference. Students will continue to use this skill later in the unit when they review writing explicit definitions for exponential and linear situations.

Step 1

- Ask students to arrange themselves in pairs or use visibly random grouping.
- Provide a minute of individual think time then ask students to discuss with their partner.

Student Task Statement

For each sequence shown, find either the common ratio or common difference. Be prepared to explain your reasoning.

1. 5, 15, 25, 35, 45, . . .
2. Starting at 10, each new term is $\frac{5}{2}$ less than the previous term.
3. 

4. $g(1) = -5, \quad g(n) = g(n-1) \cdot -2$ for $n \geq 2$
5.

n	$f(n)$
1	0
2	0.1
3	0.2
4	0.3
5	0.4

Step 2

- Display the questions for all to see throughout the discussion.
- For each problem, select students to share how they identified either the growth/decay factor or constant rate of change.

- Highlight any comments linking these sequences to linear or exponential functions. For example, students may comment that the constant rate of change would be the slope of the line containing the terms of the sequence, or they may comment that the constant rate of change is the common difference between output values from one term to the next.

DO THE MATH	PLANNING NOTES

Activity 1: Matching Recursive Definitions (*15 minutes*)

Instructional Routines: Take Turns; Discussion Supports (MLR8)	
Addressing: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.2

In this partner activity, students *Take Turns* matching a sequence to a definition. As students trade roles explaining their thinking and listening, they have opportunities to explain their reasoning and critique the reasoning of others (MP3).

One sequence and one definition do not have matches. Students are tasked with writing the corresponding match.

Step 1

- Keep students in pairs.
- Tell students that for each sequence, one partner finds its matching recursive definition and explains why they think it matches. The partner's job is to listen and make sure they agree. If they don't agree, the partners discuss until they come to an agreement. For the next sequence, the students swap roles. If necessary, demonstrate this protocol before students start working.
- Ensure that students notice that one sequence and one definition do not have matches, and they are tasked with writing the corresponding match for each.

RESPONSIVE STRATEGY

Leverage choice around perceived challenge. Invite groups of students to find matches for at least four of the sequences. Chunking this task into more manageable parts may also support students who benefit from additional processing time.

Supports accessibility for: Organization; Attention; Social-emotional skills

RESPONSIVE STRATEGY

Use this routine to support small-group discussion as students should take turns finding a match. Display the following sentence frames for all to see: "Sequence ____ and definition ____ match because . . ." and "I noticed ____, so I matched . . ." Encourage students to challenge each other when they disagree, and to press for precise mathematical language. This will help students clarify their reasoning about each match.

Discussion Supports (MLR8)

Monitoring Tip: Look for different ways students may write the recursive definition for the sequence 18, 20, 22, 24. Based on the equations in the activity, students may write using a parallel structure such as $f(1) = 18, f(n) = f(n-1) + 2, n \geq 2$. Students might write it using words or using the current term and previous term structure. Ask students whom you overheard using clear reasoning to write their definitions to share during the discussion.

***Advancing Student Thinking*:** Some students may not be sure how to begin matching terms in a sequence to a definition. Encourage them to start by picking a definition and calculating the first few terms of the sequence it represents.

Student Task Statement

Take turns with your partner to match a sequence with a recursive definition. It may help to first figure out if the sequence is arithmetic or geometric.

- For each match that you find, explain to your partner how you know it's a match.
- For each match that your partner finds, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.

There is one sequence and one definition that do not have matches. Create their corresponding match.

Sequences:

1. 3, 6, 12, 24
2. 18, 36, 72, 144
3. 3, 8, 13, 18
4. 18, 13, 8, 3
5. 18, 9, 4.5, 2.25
6. 18, 20, 22, 24

Definitions:

- $G(1) = 18, \quad G(n) = \frac{1}{2} \cdot G(n-1), \quad n \geq 2$
- $H(1) = 3, \quad H(n) = 5 \cdot H(n-1), \quad n \geq 2$
- $J(1) = 3, \quad J(n) = J(n-1) + 5, \quad n \geq 2$
- $K(1) = 18, \quad K(n) = K(n-1) - 5, \quad n \geq 2$
- $L(1) = 18, \quad L(n) = 2 \cdot L(n-1), \quad n \geq 2$
- $M(1) = 3, \quad M(n) = 2 \cdot M(n-1), \quad n \geq 2$

Step 2

- Facilitate a whole-class discussion once all groups have completed the matching. Ask, "How did you decide which definitions to match to sequence 3, 6, 12, 24 and sequence 18, 36, 72, 144 when they both involve doubling?" (They are both geometric with a growth factor of 2, but since they have different first terms, you can use those to match the sequences to M and L.)
- Next, invite previously identified students to share the recursive definition they created for sequence 18, 20, 22, 24 and their strategy for writing it.

DO THE MATH	PLANNING NOTES

Activity 2: Subscript Notation (*10 minutes*)

Instructional Routines: Notice and Wonder; Collect and Display (MLR2)	
Addressing: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.2

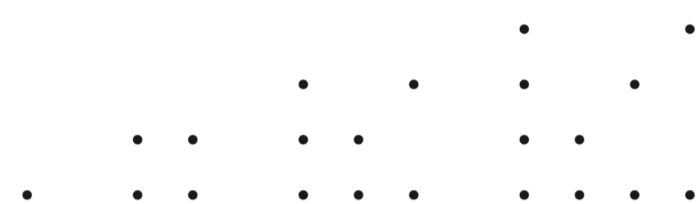
The purpose of this activity is for students to interpret a recursive definition written using subscript notation using the *Notice and Wonder* routine.

Step 1

- Display the recursive definition $a_1 = 10$, $a_n = a_{n-1} + 5; n \geq 2$ for all to see.
- Ask students to think of what they notice or wonder.
- Give students 1 minute of quiet think time and then ask students to share the things they noticed and wondered.
- Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image.
- After all responses have been recorded without commentary or editing, ask students, “Is there anything on this list that you are wondering about now?” Encourage students to respectfully disagree, ask for clarification, or point out contradicting information.
- Use students’ notices and wonderings to discuss the subscript notation. Consider asking:
  - “Where have you seen or used subscript notation?” (Possible responses: in formulas such as slope and distance; when calculating the linear regression in Desmos.)
  - “How is this similar to the function notation we have been using? How is it different?” (The forms are similar because of the use of n and $n - 1$ to identify the current term number and previous term number. Both forms also include operations such as add or multiply. The two forms are different because function notation has the term number in parentheses and the other includes the term number in the subscript.)
  - “What are the first four terms of the sequence?” (10, 15, 20, 25)
- Keep students in pairs. Encourage them to check in with their partner frequently as they work through the task.

Student Task Statement

1. For each of the following recursive definitions, calculate the first four terms of the sequence it represents and identify if it is geometric or arithmetic.
   a. $a_1 = 4$, $a_n = a_{n-1} \cdot 5$; $n \geq 2$
   b. $a = 30$, $a_n = a_{n-1} - 2$; $n \geq 2$
2. Here is a pattern where the number of dots increases with each new step. Write a recursive definition for the total number of dots d_n in Step n.



Step 1 Step 2 Step 3 Step 4

Step 2

- Use the *Collect and Display* routine to scribe the words and phrases students use as they share their reasoning during a whole-class discussion. Begin by inviting students to share how they interpreted the recursive definitions in question 1 to calculate the first four terms. While scribing, refer back to any relevant language already captured from the *Notice and Wonder* routine, and ask students for clarification of that language as needed.
- Next, invite students who have not shared to share their recursive definitions for the number of dots. Include in the discussion how the notation indicates the initial term, current term, and previous term. Use the final display of language generated by this discussion in the Lesson Debrief, if relevant.

DO THE MATH	PLANNING NOTES

Lesson Debrief (*5 minutes*)

The purpose of this lesson is for students to practice interpreting and writing recursive definitions of functions using both function notation and subscript notation.

- Refer to the display created in Lesson 4 and add the sequences for each example.
- Ask students to work with their partner to define each sequence using subscript notation.
- Invite students to share their definitions.
- Add the definitions to the poster.

Example 3 Sequence: 10, 25, 62.5, 156.25 , . . . Common Ratio (growth factor): 2.5 Recursive: $g_1 = 10$, $g_n = g_{n-1} \cdot 2.5$; $n \geq 2$	**Example 3** Sequence: 21, 29, 37, 45, . . . Common Difference (constant rate of change): 8 Recursive: $a_1 = 21$, $a_n = a_{n-1} + 8$; $n \geq 2$

PLANNING NOTES

Student Lesson Summary and Glossary

We can define sequences recursively using function notation or subscript notation. For the sequence d with terms 4, 7, 10, 13, 16, ..., the starting term is 4, and the constant rate of change is 3.

Function Notation	Subscript Notation
$d(1) = 4, d(n) = d(n-1) + 3$ for $n \geq 2$. $d(1)$ represents the first term of the sequence. $d(n)$ represents the current term in the sequence $d(n-1)$ represents the previous term in the sequence	$d_1 = 4, d_n = d_{n-1} + 3$ for $n \geq 2$. d_1 represents the first term of the sequence d_n represents the current term in the sequence d_{n-1} represents the previous term in the sequence

This type of definition tells us how to find any term, n, if we know the previous term, $n-1$. It is not as helpful in calculating terms that are far away like $d(100)$ or d_{100}. Some sequences do not have recursive definitions, but geometric and arithmetic sequences always do.

Cool-down: Represent this Sequence (*5 minutes*)

Addressing: NC.M1.F-BF.1a

Cool-down Guidance: Points to Emphasize
Spend 5 minutes at the beginning of the next lesson reviewing the cool-down with students who struggled with this and try practice problem 2 for an additional opportunity to practice.

Cool-down

Use the first five terms of sequence r to define the sequence recursively using subscript notation.

$\frac{2}{3}, 1, \frac{3}{2}, \frac{9}{4}, \frac{27}{8}$

Student Reflection:

What is something you hoped your teacher would do differently to help with your understanding of sequences? What is something that your teacher is doing well?

DO THE MATH

INDIVIDUAL STUDENT DATA

SUMMARY DATA

NEXT STEPS

TEACHER REFLECTION

What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?

What part of the lesson went really well today in terms of students' learning? What did you do that made that part go well?

Practice Problems

1. An arithmetic sequence a starts 2, 5, . . .
   a. Write a recursive definition for this sequence using subscript notation.
   b. Use your definition to find $a(6)$.
2. A geometric sequence g starts 1, 3, . . .
   a. Write a recursive definition for this sequence using subscript notation.
   b. Explain how to use the recursive definition to determine g_{30}. (Don't actually determine the value.)
3. Match each sequence with one of the recursive definitions. Note that only the part of the definition showing the relationship between the current term and the previous term is given so as not to give away the solutions.

   a. 3, 15, 75, 375
   b. 18, 6, 2, $\frac{2}{3}$
   c. 1, 2, 4, 7
   d. 17, 13, 9, 5

   1. $a(n) = \frac{1}{3} \cdot a(n-1)$
   2. $b(n) = b(n-1) - 4$
   3. $c_n = 5 \cdot c_{n-1}$
   4. $d_n = d_{n-1} + (n-1)$

4. Match each sequence with one of the recursive definitions.

   a. ...3, -9, -21, -33
   b. ...3, 5, 7, 9
   c. ...3, 10, 25, 56
   d. ...3, -9, 27, -81

   1. $a_n = a_{n-1} + 2,\ a_1 = -3$
   2. $a_n = a_{n-1} \cdot (-3),\ a_1 = -1$
   3. $a_n = a_{n-1} - 12,\ a_1 = 27$
   4. $a_n = 2a_{n-1} + n,\ a_1 - 1$

5. A geometric sequence g starts 80, 40, …
   a. Write a recursive definition for this sequence using function notation.
   b. Use your definition to make a table of values for $g(n)$ for the first six terms.
   c. Explain how to use the recursive definition to find $g(100)$. (Don't actually determine the value.)
6. Here is a table showing values of sequence p. Define p recursively using subscript notation.

n	$p(n)$
1	5,000
2	500
3	50
4	5
5	0.5

7. Write the first five terms of each sequence.

   a. $a(1) = 1$, $a(n) = 3 \cdot a(n-1)$, $n \geq 2$

   b. $b(1) = 1$, $b(n) = -2 + b(n-1)$, $n \geq 2$

   c. $c(1) = 1$, $c(n) = 2 \cdot c(n-1) + 1$, $n \geq 2$

   d. $d(1) = 1$, $d(n) = d(n-1)^2 + 1$, $n \geq 2$

   e. $f(1) = 1$, $f(n) = f(n-1) + 2n - 2$, $n \geq 2$

   (From Unit 8, Lesson 4)

8. A sequence has $f(1) = 120, f(2) = 60$.

   a. Determine the next two terms if it is an arithmetic sequence, then write a recursive definition that matches the sequence in the form $f(1) = 120$, $f(n) = f(n-1)+$______ for $n \geq 2$.

   b. Determine the next two terms if it is a geometric sequence, then write a recursive definition that matches the sequence in the form $f(1) = 120$, $f(n) =$______ $\cdot f(n-1)$ for $n \geq 2$.

   (From Unit 8, Lesson 4)

9. One hour after an antibiotic is administered, a bacteria population is 1,000,000. Each subsequent hour, it decreases by a factor of $\frac{1}{2}$.

   a. Complete the table with the bacteria population at the given times.

   b. Do the bacteria populations make a geometric sequence? Explain how you know.

   (From Unit 8, Lesson 2)

Number of hours	Population
1	1,000,000
2	
3	
4	
5	
6	

10. Rewrite each expression in standard form.

    a. $(x+3)(x-3)$ b. $(7+x)(x-7)$ c. $(2x-5)(2x+5)$ d. $(x+\frac{1}{8})(x-\frac{1}{8})$

    (From Unit 7)

11. Consider the equation $y = 2x(6-x)$.

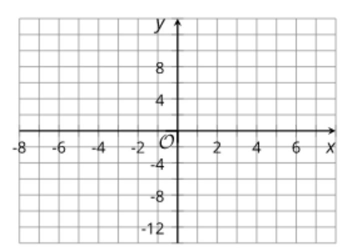
    a. What are the x-intercepts of the graph of this equation? Explain how you know.

    b. What is the x-coordinate of the vertex of the graph of this equation? Explain how you know.

    c. What is the y-coordinate of the vertex? Show your reasoning.

    d. Sketch the graph of this equation.

    (From Unit 7)



12. (*Technology required*). A moth population, p, is modeled by the equation $p = 500,000 \cdot (\frac{1}{2})^w$, where w is the number of weeks since the population was first measured.

    a. What was the moth population when it was first measured?

    b. What was the moth population after 1 week? What about 1.5 weeks?

    c. Use technology to graph the population over time and find out when it falls below 10,000.

    (From Unit 6)

Lesson 6: The n^{th} Term

PREPARATION

Lesson Goals	Learning Targets
• Interpret an equation for the n^{th} term of a sequence. • Justify (orally and in writing) why different equations can represent the same sequence.	• I can interpret an equation for the n^{th} term of a sequence. • I can explain why different equations can represent the same sequence.

Lesson Narrative

The goal of this lesson is for students to understand that how an equation is written to represent a function depends on how the domain of a function is identified. With sequences, it is common to start at either $f(1)$ or $f(0)$. So far in this unit, the first term has typically been written as $f(1)$. The exception has been when $n = 1$ is confusing given the context, which is the case when the number of pieces of paper depends on the number of cuts. This lesson gives students a chance to study the effect this choice has when writing an equation to define a sequence. It is also meant to help students review how to write equations of linear and exponential functions by using a table to express regularity in repeated reasoning (MP8). In the following lessons, students will write equations for these types of functions in various contexts.

Prior to this lesson, students focused on defining sequences recursively using function notation and subscript notation. In this lesson, students will study equations representing functions that are known as **explicit or closed-form definitions**. A closed-form definition is one where the value of the n^{th} term is determined from just the term number. This type of equation is one students are familiar with from their earlier work with linear and exponential equations.

A focus of this lesson is using precise language to explain patterns and understand how a function can be represented by two different equations (MP6).

How might the learning activities in this lesson build on your students' strengths? How can you use these activities to highlight student success and growth?

Focus and Coherence

Building On	Addressing
NC.8.F.4: Analyze functions that model linear relationships. • Understand that a linear relationship can be generalized by $y = mx + b$. • Write an equation in slope-intercept form to model a linear relationship by determining the rate of change and the initial value, given at least two (x, y) values or a graph. • Construct a graph of a linear relationship given an equation in slope-intercept form. • Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of the slope and y-intercept of its graph or a table of values. **NC.M1.F-LE.1:** Identify situations that can be modeled with linear and exponential functions, and justify the most appropriate model for a situation based on the rate of change over equal intervals.	**NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). **NC.M1.F-BF.2:** Translate between explicit and recursive forms of arithmetic and geometric sequences and use both to model situations.

Agenda, Materials, and Preparation

Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate technology to solve problems. It is ideal if each student has accessibility to their own device.

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
  - *Optional*: Pieces of Paper grid (print 1 copy per student); if using, prepare 1 pair of scissors for every 2 students.
- **Activity 2** (*Optional, 15 minutes*)
- **Activity 3** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U8.L6 Cool-down (print 1 copy per student)

LESSON

Bridge (*Optional, 5 minutes*)

Building On: NC.8.F.4

The purpose of the bridge is to provide a review of writing linear equations given the constant rate of change and a point. In this case, the two points have been selected to connect to later work defining sequences with a linear function. In question 1, the point is similar to when the initial term of the sequence is a_1. The second question is similar to when the initial term is a_0.

Student Task Statement

1. Write a linear equation that has a constant rate of change of 6 and passes through the point $(1, 3)$.
2. Write a linear equation that has a constant rate of change of -2 and passes through the point $(0, 10)$.

DO THE MATH

PLANNING NOTES

Warm-up: Repeated Operations (*5 minutes*)

Instructional Routines: Which One Doesn't Belong?	
Building On: NC.M1.F-BF.1a	**Building Towards:** NC.M1.F-BF.2

This *Which One Doesn't Belong?* warm-up prompts students to compare four expressions. It gives students a reason to use language precisely (MP6). It gives the teacher an opportunity to hear how students use terminology and talk about characteristics of the items in comparison to one another.

The purpose of this activity is to remind students that repeated addition can be represented with multiplication and repeated multiplication can be represented with an exponent, which will help them make sense of some of the different ways equations for sequences can be written.

Step 1

- Ask students to arrange themselves in small groups or use visibly random grouping.
- Display the expressions for all to see.
- Give students 1 minute of quiet think time and then time to share their thinking with their small group. In their small groups, ask each student to share their reasoning why a particular item does not belong. Then groups should collaboratively find a reason for any item that was not shared.

Student Task Statement

Which one doesn't belong? Explain your reasoning.

a. $5+2+2+2+2+2+2$	b. $5+2(6)$
c. $5 \cdot 2^6$	d. $5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

Step 2

- Ask each group to share one reason why a particular item does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given are correct.
- If it doesn't come up during the discussion, highlight the fact that "a" and "b" are equivalent expressions, and "c" and "d" are equivalent expressions.

DO THE MATH	PLANNING NOTES

Activity 1: More Pieces of Paper (*15 minutes*)

Building On: NC.M1.F-LE.1	**Addressing:** NC.M1.F-BF.1a; NC.M1.F-BF.2

In this activity students return to the 8-inch-by-10-inch grid from an earlier lesson. This time, the goal is to work with formulas that give non-recursive definitions of two different sequences, one geometric and one arithmetic, based on the grid being cut up in different ways. Students express regularity in repeated reasoning (MP8) by using their understanding of how the values of specific terms are calculated before explaining or expressing how the n^{th} term is calculated. Making graphing technology available gives students an opportunity to choose appropriate tools strategically (MP5).

Step 1

- Ask students to arrange themselves in pairs or use visibly random grouping.
- If using, distribute scissors and two copies of the Pieces of Paper grid blackline master to each group.
- Ask students to recall the paper cutting activity from an earlier lesson. In it, Clare takes a sheet of paper that is 8 inches by 10 inches, cuts the paper in half, stacks the pieces, cuts the pieces in half, then stacks them, etc. We can let $C(n)$ be the area, in square inches, of each piece based on the number of cuts n. Display the table for all to see.

n (number of cuts)	$C(n)$ (area in square inches of each piece)
0	80
1	40
2	20
3	10
4	5

  - Depending on time, either invite students to share their observations about $C(n)$ or tell students that it is a geometric sequence with common ratio $\frac{1}{2}$.
  - Say, "This sequence starts with $n = 0$ since we start with a piece of paper with 0 cuts. How can we write a formula that is a recursive definition for $C(n)$?" Provide students a minute to discuss with their partner.
  - Select students to share their definitions paying particular attention to the starting term, $C(0)$ or C_0, and that $n \geq 1$ is used. ($C(0) = 80$, $C(n) = C(n-1) \cdot \frac{1}{2}$, $n \geq 1$ or $C_0 = 80$, $C_n = C_{n-1} \cdot \frac{1}{2}$, $n \geq 1$.)
  - An important takeaway from looking at this recursive definition is that the domain of a sequence is something that should be based on the situation and that sometimes starting with $n = 1$ doesn't make sense. Luckily, we can use the function notation or subscript notation to make clear that we are starting with $n = 0$ by beginning with $C(0)$ or C_0 and then we change $n \geq 2$ to $n \geq 1$ to match.
- Ask students to work together on problems 1–3.

Advancing Student Thinking: Students who have trouble visualizing what's happening to the paper in each sequence may benefit from drawing the paper at each step and labeling it with dimensions, or cutting paper themselves and calculating the areas. In particular, if students don't see why Kiran removes 8 square inches each time, encourage them to write down the dimensions of the paper for the first few steps and calculate each area (and draw the paper at each step if needed).

Student Task Statement

1. Clare takes a piece of paper with length 8 inches and width 10 inches and cuts it in half. Then she cuts it in half again, and again...
   a. Instead of writing a recursive definition, Clare writes $C(n) = 80 \cdot (\frac{1}{2})^n$, where $C(n)$ is the area, in square inches, of the paper after n cuts. Explain where the different terms in her expression came from.
   b. Approximately what is the area of the paper after 10 cuts? What is the area after 25 cuts?
2. Kiran has a piece of paper with length 8 inches and width 10 inches. He cuts off one end of the paper, making a strip that is 1 inch by 8 inches. Then he does it again, and again...
   a. Complete the table for the area of Kiran's paper, $K(n)$, in square inches, after n cuts.
   b. Kiran says the area after 6 cuts, in square inches, is $80 - 8 \cdot 6$. Explain where the different terms in his expression came from.
   c. Write a definition for $K(n)$ that is not recursive.
3. Which is greater, $K(6)$ for Kiran's function or $C(6)$ for Clare's function?

n	$K(n)$
0	80
1	
2	$80 - 8 - 8 = 80 - 8(2) = 64$
3	
4	
5	

Step 2

- Display the two definitions from this task for all to see: $C(n) = 80 \cdot (\frac{1}{2})^n$ and $K(n) = 80 - 8n$, as well as the table from Step 1
- Begin the discussion by saying, "We can find the area of Claire's paper after she makes 3 cuts by finding $80 \cdot (\frac{1}{2})^3$. Why does that work?" (The original area of the paper is 80 square inches. Each time Clare makes a cut, the new area is half the old area. Multiplying 80 by $\frac{1}{2}$ 3 times is the same as multiplying by $(\frac{1}{2})^3$.)
- "We can find the area of the Kiran's paper after four cuts by finding $80 - 8 \cdot 4$. Why does that work?" (The original area of the paper is 80 square inches. With each cut, Kiran removes 8 square inches. $80 - 8 - 8 - 8 - 8$ is the same as $80 - 8 \cdot 4$.)
- Then ask, "How can you tell which of the functions defines a geometric sequence and which defines an arithmetic sequence?" (A geometric sequence has a common ratio between terms which, in this case, is the decay factor. That means $C(n)$ must represent a geometric sequence, since each term is half the value of the previous term. An arithmetic sequence has a common difference, which is the constant rate of change. That means $K(n)$ must represent an arithmetic sequence, since each term is 8 less than the previous term.)

RESPONSIVE STRATEGY
Break the class into small group discussion groups and then invite a representative from each group to report back to the whole class.

Supports accessibility for: Language; Social-emotional skills; Attention

- Tell students that $C(n)$ and $K(n)$ are examples of defining a sequence by the n^{th} term and are known as a "closed-form or explicit" definition. Ask students, "What type of function is $C(n)$? $K(n)$? ($C(n)$ is an exponential function, and $K(n)$ is a linear function.)
- Tell students that if asked to represent a sequence with an equation for the n^{th} term, then they are being asked for this type of equation and not to define the sequence recursively.
- Ask students to talk to a partner about whether they would rather use an explicit definition for finding the area of the paper after 25 cuts or a recursive definition. Have a few students share with the whole class. (The explicit definition is much quicker since using the recursive definition requires performing 25 calculations.)
- Lastly, if time allows, ask students to calculate which is larger, $K(10)$ or $C(10)$. ($C(10)$ is larger, since $K(10) = 0$ and $C(10) > 0$, or there can't be a comparison because $K(10)$ does not exist, since a tenth cut is not possible.) Select students to share their calculations. If no student points out that $K(10)$ does not exist due to the constraints of the context when n is the number of cuts, make sure to bring this point up.

DO THE MATH	PLANNING NOTES

Activity 2: A Sierpinski Triangle (*Optional, 15 minutes*)

Instructional Routines: Notice and Wonder; Stronger and Clearer Each Time (MLR1)
Addressing: NC.M1.F-BF.2

This activity is optional. The goal of this activity is for students to understand that the way an equation is written to define a function depends on how the domain of the function is interpreted. In the previous activity, starting the sequence C with $C(0)$ or C_0 made sense since n was defined as the number of cuts while $C(n)$ or C_n is the area of a paper square after n cuts. In this activity, students consider two equations written for a sequence represented by a visual pattern using the *Notice and Wonder* routine. One of the equations assumes the sequence starts at Step 0, and the other assumes the sequence starts at Step 1.

To help all students understand that both equations generate the same sequence and correctly represent the visual pattern, during the activity and whole-class discussion, students should be encouraged to use precise language as they explain why an equation is valid (MP6). Confusion is likely to arise unless students are clear about the definition for n each equation is assuming. Some students may find the idea that both equations are equally valid challenging, so it is important to take time during the discussion for these students to understand that part of mathematics is recognizing what assumptions you are making and addressing them appropriately.

Step 1

- Display the triangles without the explanation and give students 1 minute to share anything they notice or wonder.
- Keep students in pairs. Provide students 5 minutes to work on questions 1 and 2.

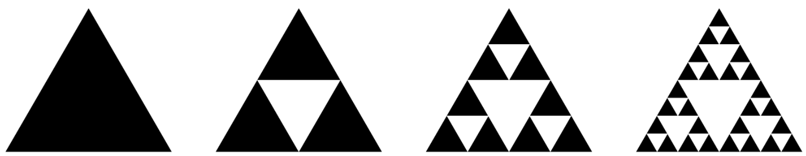
Monitoring Tip: Monitor for students with clear definitions to select to share during the whole-class discussion. In particular, highlight responses from students that create different representations, such as tables, to see that the two equations result in the same list of numbers as their recursive definitions.

In addition, listen for students that provide reasons why Andre's equation is correct, Lin's equation is correct, or both equations are correct. Identify students and let them know that they may be asked to share later. Include at least one student who does not typically volunteer.

***Advancing Student Thinking*:** Students may assume that at least one person has to be wrong because their equations aren't equivalent. Encourage them to make a table for each equation that starts with the first value for n according to the definitions.

Student Task Statement

A Sierpinski triangle can be created by starting with an equilateral triangle, breaking the triangle into four congruent equilateral triangles, and then removing the middle triangle. Starting from a single black equilateral triangle:



1. Let S be the number of black triangles in Step n. Define $S(n)$ recursively.
2. Andre and Lin are asked to write an equation for S that isn't recursive. Andre writes $S(n) = 3^n$ for $n \geq 0$, while Lin writes $S(n) = 3^{n-1}$ for $n \geq 1$. Whose equation do you think is correct? Explain or show your reasoning.

Are You Ready For More?

Here is a geometric sequence. Find the missing terms.

3, ____, 6, ____, 12, ____, 24

Step 2

The goal of this discussion is for students to understand why Andre's and Lin's equations are both representations of the visual pattern due to the interpretation of n.

- Display the visual pattern for all to see.
- Invite previously selected students to briefly share their reasoning about why either Andre or Lin is correct. If students still think starting with 0 is unusual after the share out, give some more time for students to reason about Andre's point of view. (For example, Andre could be thinking about n as the breaking apart phase; for the solid black triangle, that has happened 0 times.)
- Conclude the discussion by telling students that identifying an appropriate domain for a function is partly dependent on the situation and partly dependent on how they see the relationship. There are often many correct equations that represent a function. An important takeaway for students is that when they write an equation to represent a situation, they need to be clear what domain they have identified so other people can correctly interpret what they've done.

Step 3

- Use the *Stronger and Clearer Each Time* routine to provide students with an additional opportunity to consolidate their understanding and clarify their explanations for question 2 through conversation.
- Give students time to meet with two or three partners (1–2 minutes with each partner) to share their initial response to question 2. Invite listeners to ask questions, add ideas, and press for details and additional mathematical language.
- Then give students 2 minutes to produce a revised written response that is stronger and clearer than their initial writing. Encourage them to include good ideas they got from their partners.

DO THE MATH	PLANNING NOTES

Activity 3: Recursive and Explicit Definitions (*10 minutes*)

Instructional Routine: Take Turns
Addressing: NC.M1.F-BF.2

The purpose of this activity is for students to translate between recursive and explicit forms of arithmetic and geometric sequences.

Step 1

- Keep students in pairs. Have students *Take Turns* filling in one of the blanks in the table for question 1, allowing students to choose a blank still to be filled each time, regardless of order. This will allow students to begin where they are most comfortable.
- Have students complete question 2 collaboratively with their partners.

Monitoring Tip: Look for how students approach translating from one definition to another. Students may:
- generate the first few terms of the sequence and use the terms to write the definition
- identify the common difference/ratio and the initial value to write it using the structure of definition

***Advancing Student Thinking*:** Some students may be unsure on how to start. Suggest they use the definition given to generate the first few terms of the sequence and then write the other definition.

Student Task Statement

1. Complete the following table. In each case, let the initial value be represented by a_1 or $a(1)$.

Sequence	Recursive Definition	Explicit Definition
a. $9, 15, 21, 27, ...$		
b.	$a_1 = 12$, $a_n = a_{n-1} \cdot 3$, $n \geq 2$	
c.		$a(n) = 120(\frac{1}{2})^n$ for $n \geq 1$
d.		$a(n) = 14 - 9n$ for $n \geq 1$

2. Noah and Han are writing an explicit definition for the sequence defined by $a_1 = 16$, $a_n = a_{n-1} + 7$, $n \geq 2$. Noah wrote $a(n) = 16 + 7n$, and Han wrote $a(n) = 9 + 7n$.
   a. Decide which explicit definition is correct and explain how you know.
   b. For the explicit definition that is incorrect, explain the mistake and how to correct it.

Are You Ready For More?

Another way to write an explicit definition for Noah and Han's sequence is $a(n) = 16 + (7n - 1)$, $n \geq 1$.

1. Use this definition to calculate the first, second, and third term of the sequence. Why do we need to write "$n - 1$" instead of "n"?
2. Use point-slope form to write an equation for a line that passes through the point ($(1, 16)$and has slope 7. What do you notice?

Step 2

- Invite previously selected students to share how they wrote their definitions.
- Facilitate a whole-class discussion. Possible points to include:
  - Ask, "How do you determine if the explicit definition should be a linear or an exponential function?" (An arithmetic sequence, which has a common difference, can be defined by a linear function. A geometric sequence, which has a common ratio, can be defined by an exponential function.)
  - Display the two explicit definitions for 1b and discuss how they are similar and how they are different. (Using the rules $a(n) = 12(3)^{n-1}$, $n \geq 1$ and $a(n) = 4(3)^n$, $n \geq 1$: Both rules have 3 as the base of an exponent, but they have different exponents (n and n-1). In one rule, you can see the initial value, 12, but in the other rule you can't. For the first rule, you use the exponent $n - 1$ because $n^{1-1} = 3^0 = 1$, making the first term 12. If you want to use n as the exponent, use 4 instead of 12, because $4(3)^1 = 4 \cdot 3 = 12$.)

DO THE MATH	PLANNING NOTES

Lesson Debrief (*5 minutes*)

The purpose of this lesson is to make connections between the recursive and explicit definitions of arithmetic and geometric sequences, and to consider the desired domain when writing these definitions.

Display these two sequences from an earlier lesson for all to see:

B: 1, 2, 4, 8, 16, . . .

D: 20, 13, 6, -1, -8, . . .

- Invite students to pick one of the sequences and write an equation for the n^{th} term. If students are unsure where to start, remind them of their work writing equations for linear and exponential functions done earlier in this course.
- Arithmetic and geometric sequences are just special types of these functions with a restricted domain. After some work time, select students to share their equations, making sure they clearly state whether they assumed the sequence started with $n = 0$ or $n = 1$. For example, an equation for D could be $D(n) = 20 - 7(n)$ for $n \geq 0$ or it could be $D(n) = 20 - 7(n-1)$ for $n \geq 1$.
- After students agree on some equations, ask them to find a recursive definition for D. (For example: $D(1) = 20$, $D(n) = D(n-1)$, $n \geq 2$) Then ask students whether they would rather use the explicit or the recursive definition to find $D(100)$..

PLANNING NOTES

Student Lesson Summary and Glossary

Here is an arithmetic sequence f: 6, 10, 14, 18, 22, ...

In this sequence, each term is 4 more than the previous term. One recursive definition of this sequence is $f_1 = 6$, $f_n = f_{n-1} + 4$ for $n \geq 2$. We could also write $f_0 = 6$, $f_n = f_{n-1} + 4$ for $n \geq 1$ since it generates the same sequence. Neither of these definitions is better than the other; we just have to remember how we chose to define the “first term” of the sequence: f_1 or f_0. Let's use f_1 for now.

While defining a sequence recursively works to calculate the current term from the previous, if we wanted to calculate, say, f_{100}, it would mean calculating all the terms up to f_{99} to get there! Let's think of a better way.

Since we know that each term has an increasing number of fours, we could write the terms of f organized in a table like the one shown here.

n	$f(n)$
1	$6+0=6+4(0)=6$
2	$6+4=6+4(1)=10$
3	$6+4+4=6+4(2)=14$
4	$6+4+4+4=6+4(3)=18$
5	$6+4+4+4+4=6+4(4)=22$

Looking carefully at the pattern in the table, we can say that for the n^{th} term $f(n)=6+4(n-1)$ for $n \geq 1$. This is sometimes called an **explicit** or **closed-form definition** of a sequence, but it's really just a way to calculate the value of the n^{th} term without having to calculate all the terms that came before it. Need to know $f(100)$? Just compute $6+4(100-1)$. Defining an arithmetic sequence this way takes advantage of the fact that this type of sequence is a linear function with a starting value (in this case, 6) and rate of change (in this case, 4). If we had decided to start the sequence at $n=0$ so that $f(0)=6$, we would have written the equation for the n^{th} term as $f(n)=6+4(n)$ for $n \geq 0$.

Geometric sequences behave the same way, but with repeated multiplication. The geometric sequence g: 3, 15, 75, 375, . . . can be written as 3, $3 \cdot 5$, $3 \cdot 5 \cdot 5$, $3 \cdot 5 \cdot 5 \cdot 5$, ... This means if $g(0)=3$, we can define the n^{th} term directly as $g(n)=3 \cdot 5^n$.

Explicit (or closed-form) definition of a sequence: A way of writing a rule for the terms of sequence that does not depend on the terms that came before. For example, the sequence a: 10, 8, 6, 4… has explicit definition $a(n)=10-2(n-1)$, where $n \geq 1$.

Cool-down: Different Types of Equations (*5 minutes*)

Addressing: NC.M1.F-BF.2

Cool-down Guidance: Press Pause
At this point, students need to be able to write recursive definitions of a given sequence. If students continue to struggle, select examples of cool-downs from this unit to highlight and clarify misconceptions. Practice problems 1, 2, and 3 from this lesson all provide opportunities for practice and additional formative assessment.

Cool-down

A sequence is defined by $f(0)=-3$, $f(n)=f(n-1)+0.5$ for $n \geq 1$. Diego writes an equation for the n^{th} term of the sequence as $f(n)=-3+0.5n$ for $n \geq 0$. Is he correct? Explain how you know.

Student Reflection:

Which parts of sequences do you understand really well? What do you still need more help understanding?

DO THE MATH

INDIVIDUAL STUDENT DATA

SUMMARY DATA

NEXT STEPS

TEACHER REFLECTION

What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?

What unfinished learning or misunderstandings do your students have about sequences? How did you leverage those misconceptions in a positive way to further the understanding of the class?

Practice Problems

1. A sequence is defined by $f(0) = -20$, $f(n) = f(n-1) - 5$ for $n \geq 1$.

   a. Explain why $f(1) = -20 - 5$.

   b. Explain why $f(3) = -20 - 5 - 5 - 5$.

   c. Complete the expression: $f(10) = -20 -$________. Explain your reasoning.

2. A sequence is defined by $f(0) = -4$, $f(n) = f(n-1) - 2$ for $n \geq 1$. Write a definition for the n^{th} term of the sequence.

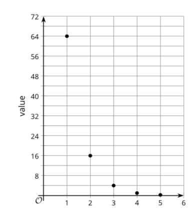
3. A sequence is defined by the explicit function $f(n) = 3n + 5$.

   a. Complete the table to show the first four terms of the sequence.

n	0	1	2	3
$f(n)$				

   b. Write the recursive equation f_n that defines the sequence. Explain your reasoning.

4. Here is the graph of a sequence:

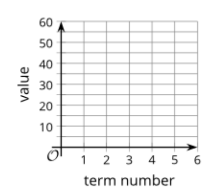


   a. Is this sequence arithmetic or geometric? Explain how you know.

   b. List at least the first five terms of the sequence.

   c. Write a recursive definition of the sequence.

5. Here is the recursive definition of a sequence: $f(1) = 3$, $f(n) = 2 \cdot f(n-1)$ for $n \geq 2$.



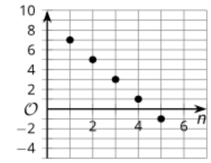
   a. Find the first five terms of the sequence.

   b. Graph the value of the term as a function of the term number.

   c. Is the sequence arithmetic, geometric, or neither? Explain how you know.

   (From Unit 8, Lesson 5)

6. Here is a graph of sequence M. Define M recursively using function notation.



(From Unit 8, Lesson 5)

7. Write the first five terms of each sequence. Determine whether each sequence is arithmetic, geometric, or neither.

   a. $a(1) = 5$, $a(n) = a(n-1) + 3$ for $n \geq 2$.

   b. $b(1) = 1$, $b(n) = 3 \cdot b(n-1)$ for $n \geq 2$.

   c. $c(1) = 3$, $c(n) = -c(n-1) + 1$ for $n \geq 2$.

   d. $d(1) = 5$, $d(n) = d(n-1) + n$ for $n \geq 2$.

(From Unit 8, Lesson 4)

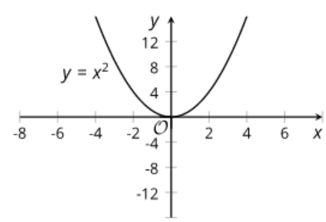
8. Four students solved the equation $x^2 + 225 = 0$. Their work is shown here. Only one student solved it correctly.

Student A:	Student B:	Student C:	Student D:
$x^2 + 225 = 0$	$x^2 + 225 = 0$	$x^2 + 225 = 0$	$x^2 + 225 = 0$
$x^2 = -225$	$x^2 = -225$	$(x-15)(x+15) = 0$	$x^2 = 225$
$x = 15$ or $x = -15$	No solutions	$x = 15$ or $x = -15$	$x = 15$ or $x = -15$

Determine which student solved the equation correctly. For each of the incorrect solutions, explain the mistake.

(From Unit 7)

9. Here is a graph that represents $y = x^2$.



   a. Describe what would happen to the graph if the original version was changed to:

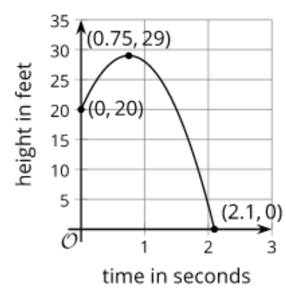
      i. $y = \frac{1}{2}x^2$

      ii. $y = x^2 - 8$

   b. Graph the equation $y = \frac{1}{2}x^2 - 8$ on the same coordinate plane as $y = x^2$.

(From Unit 7)

10. Clare throws a rock into the lake. The graph shows the rock's height above the water, in feet, as a function of time in seconds.



Select **all** the statements that describe this situation.

   a. The vertex of the graph is $(0.75, 29)$.
   b. The y-intercept of the graph is $(2.1, 0)$.
   c. Clare dropped the rock into the lake without throwing it upwards.
   d. The maximum height of the rock is about 20 feet.
   e. The rock hits the surface of the water after about 2.1 seconds.
   f. Clare tossed the rock up into their air from a point 20 feet above the water.

(From Unit 7)

11. (*Technology required*). Two objects are launched into the air. In both functions, t is seconds after launch.

- The height, in feet, of object A is given by the equation $f(t) = 4 + 32t - 16t^2$.
- The height, in feet, of object B is given by the equation $g(t) = 2.5 + 40t - 16t^2$.

Use technology to graph each function in the same graphing window.

a. What is the maximum height of each object?

b. Which object hits the ground first? Explain how you know.

(From Unit 7)

12. Jada and Lin were born on the same day to different families. Each family has grandparents that want to contribute money to an account for them to use when they complete high school.

Lin's grandparents have chosen to contribute \$500 on the day he is born and then \$100 per year until he is 18. Let $f(x)$ represent the amount of money in Lin's account after x years. Jada's grandparents have chosen to contribute \$650 on the day he is born into an account that accrues interest, but they will never add any more money. The amount of money in Jada's account can be seen in the table to the right:

x	$g(x)$
0	650
2	716.625
4	790.07906
6	871.06217
8	960.34604
10	1058.7815
12	1167.3066
14	1286.9555
16	1418.8685
18	1564.3025

Compare and interpret the following features of each baby's account:

a. Vertical intercepts

b. $f(8)$ and $g(8)$

c. Average rate of change on the interval 0 to 4

d. Amount of money in the account when each person turns 18

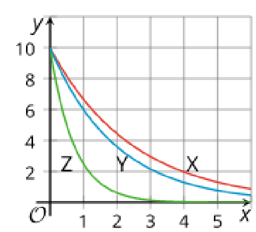
(From Unit 7)

13. Here are the graphs of three equations.

Match each graph with the appropriate equation.

a. $y = 10(\frac{2}{3})^x$
b. $y = 10(\frac{1}{4})^x$
c. $y = 10(\frac{3}{5})^x$

1. Graph X
2. Graph Y
3. Graph Z



(From Unit 6)

Lesson 7: Situations and Sequences

PREPARATION

Lesson Goals	Learning Targets
• Identify restrictions on the domain of a sequence based on context. • Use sequences to model situations.	• I can identify the domain of a sequence. • I can represent situations with sequences.

Lesson Narrative

In this lesson, students practice modeling situations using different types of equations for sequences. They then use their equations and other representations to answer questions about the situation, translating between the situations and their representations (MP2). This isn’t meant to be the full modeling cycle but rather a focus on some practices that students must attend to while modeling (MP4). In this lesson, students describe a reasonable domain for a function representing a context where an unrestricted domain does not make sense.

Encourage students to represent the sequences in multiple ways to help them make sense of the patterns they see. Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate technology to solve problems.

Can students benefit from learning or creating another method for a specific skill here, or does the standard focus on a particular method?

Focus and Coherence

Building On	Addressing
NC.M1.F-LE.1: Identify situations that can be modeled with linear and exponential functions, and justify the most appropriate model for a situation based on the rate of change over equal intervals. **NC.M1.F-LE.3:** Compare the end behavior of linear, exponential, and quadratic functions using graphs and tables to show that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically. **NC.M1.F-IF.8:** Use equivalent expressions to reveal and explain different properties of a function.	**NC.M1.F-IF.5:** Interpret a function in terms of the context by relating its domain and range to its graph and, where applicable, to the quantitative relationship it describes. **NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities. **a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). **NC.M1.F-BF.2:** Translate between explicit and recursive forms of arithmetic and geometric sequences and use both to model situations.

Agenda, Materials, and Preparation

- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*10 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U8.L7 Cool-down (print 1 copy per student)

LESSON

Warm-up: Describing Growth (*5 minutes*)

Instructional Routine: Poll the Class	
Building On: NC.M1.F-IF.8	

This warm-up is a review of the relationship between percent change and the growth factor of an exponential function.

Step 1

- Provide some quiet time for students to think through the questions.
- Ask students to share a response or guess that they can justify.
- *Poll the Class* for how many agree with each possibility suggested. Then ask students to justify, with at least one justification, for a common incorrect response. Ask the students to compare the reasoning and try to arrive at a class consensus before reminding students of the difference between the growth rate and the growth factor.

Student Task Statement

1. Here is a geometric sequence: 16, 24, 36, 54, 81

   What is the growth factor?

2. One way to describe its growth is to say it’s growing by __% each time. What number goes in the blank for the sequence 16, 24, 36, 54, 81? Be prepared to explain your reasoning.

Step 2

- Facilitate a whole-class discussion with the goal to check that students understand the difference between growth rate, r, and growth factor, $r+1$, when talking about a sequence.
  - Ask a student to share how they calculated the growth factor and why the number they chose for the percent makes sense.
  - If needed, remind students that 1.5 is known as the “growth factor” for this function, and 0.5 is known as the “growth rate.” Growth rate is often expressed as a percentage, so 50%. The result of increasing a quantity by 50% can be calculated by multiplying it by 1.5.
- Use the following questions to guide the class discussion.
  - “How can you define the n^{th} term of the sequence?” ($f(n)=16(1.5)^n$ for $n\geq 0$.)
  - “Where in that function do you see the growth rate and growth factor?” (The growth factor is the value being repeatedly multiplied for different values of n. The growth rate is the percent change between one term and the next, so in this case, it is the growth factor minus 1, which is 0.5.)

- If students need additional practice differentiating between growth rate, r, and growth factor, $r+1$, display the following questions for all to see and ask students to calculate the growth factor and growth rate for each.
    1. 64, 80, 100, 125 (1.25 and 25%)
    2. 64, 112, 196, 343 (1.75 and 75%)
    3. 64, 128, 256, 512 (2 and 100%)
    4. 125, 100, 80, 64 (0.8 and 20% decrease)
- After a brief work time, select students to share how they calculated their solutions.

DO THE MATH	PLANNING NOTES

Activity 1: Finding Population Patterns (*15 minutes*)

This is the first of two activities where students define sequences with equations and use their equations to answer questions about the context (MP2). The population values were purposefully chosen in order for students to focus on creating representations (like tables and graphs) and not on calculating "best fit."

Making spreadsheet and graphing technology available gives students an opportunity to choose appropriate tools strategically (MP5).

Instructional Routine: Compare and Connect (MLR7)	
Building On: NC.M1.F-LE.1; NC.M1.F-LE.3; NC.M1.F-IF.8	**Addressing:** NC.M1.F-BF.1a; NC.M1.F-BF.2

Step 1

- Ask students to arrange themselves in small groups or use visibly random grouping.
- Display the table for all to see.
- Ask students, "What are some ways you could figure out if the sequences represented by the populations are arithmetic, geometric, or neither?" (Using the terms, perform calculations to check for a common difference (arithmetic) or common ratio (geometric). List strategies but not solutions, making sure all students have a starting point before independent work time.)

Years since 1990	Population A	Population B
0	23,000	3,125
1	29,000	3,750
2	35,000	4,500
3	41,000	5,400
4	47,000	6,480

- Give students quiet work time and then time to share their work in their groups. Instruct groups to create a visual display of their reasoning for the last question about population B overtaking population A.

Advancing Student Thinking: If students incorrectly identify the type of sequence from the table, encourage these students to graph the points or, if they have already made an equation, check that their equation generates the sequence correctly.

Monitoring Tip: As students are working on their equations for population A and population B, take note of the strategy used. Look for students who:

- extend the table to include additional rows
- use graphing technology to create a graph
- arrive at an explicit definition
- arrive at a recursive definition

Student Task Statement

The table shows two animal populations growing over time.

1. Are the sequences represented by population A and population B arithmetic or geometric? Explain how you know.
2. Write a definition for population A.
3. Write a definition for population B.
4. Does population B ever overtake population A? If so, when? Explain how you know.

Years since 1990	Population A	Population B
0	23,000	3,125
1	29,000	3,750
2	35,000	4,500
3	41,000	54,000

Step 2

- Select groups to share displays of the reasoning they used to answer question 4. Check that their displays include their recursive or n^{th} term definitions of populations A and B, and use information from your monitoring to have students share in the sequence noted in the Monitoring Tip.
- Use the *Compare and Connect* routine to have selected groups present their displays, and as they share, prompt the class to engage with the ways in which each representation shows different features of the sequence: for example, the growth factor or rate of change. When students are comparing the approaches and representations used, prompt for and amplify comments both about what is easier to see and understand in each as well as about what might make each more complete or clear.
- As groups share how they determined whether population B would ever overtake population A, call students' attention to the different ways that they reasoned about finding the year and to the different ways these methods are made visible in each representation.

DO THE MATH

PLANNING NOTES

Activity 2: Take the Cake! (*10 minutes*)

Instructional Routines: Aspects of Mathematical Modeling; Co-Craft Questions (MLR5) - Responsive Strategy
Addressing: NC.M1.F-IF.5; NC.M1.F-BF.1a; NC.M1.F-BF.2

The purpose of this activity is for students to write definitions for a geometric sequence given by a situation. Students must translate from the description to mathematical representations (MP2). The context puts some limitations on the domain, and students engage in *Aspects of Mathematical Modeling* (MP4) by articulating why. For example, there is a difference between which numbers we can substitute for n in an equation representing the amount of cake left after the n^{th} person takes their piece and which numbers make sense to substitute for n. There comes a point where the amount of cake is too small to reasonably expect a knife to slice or a scale to measure $\frac{1}{3}$ by weight.

Step 1

- Allow students to continue working in their small groups.
- Display the task statement for all to see and tell students to read the description of the situation. Then ask, "Is there still cake left after three people each take some cake?"
- After a brief quiet work time, select students who reply "yes" to explain their thinking. If any of them created a diagram, such as a circle split into smaller and smaller pieces, display it for all to see. Ensure students understand that each new person takes $\frac{1}{3}$ of what is left, not $\frac{1}{3}$ of the original amount.
- Provide groups with work time to complete the task.

Student Task Statement

A large cake is in a room. The first person who comes in takes $\frac{1}{3}$ of the cake. Then a second person takes $\frac{1}{3}$ of what is left. Then a third person takes $\frac{1}{3}$ of what is left. And so on.

1. Complete the table for $C(n)$, the fraction of the original cake left after n people take some.
2. Write two definitions for C: one recursive and one non-recursive.
3. Construct a graph of the situation, giving special attention to the possible input values.
4. What is a reasonable domain for this function? Be prepared to explain your reasoning.

n	$C(n)$
0	
1	$\frac{2}{3}$
2	
3	
4	

Step 2

- Ask students to share their definitions, recording for all to see.
- Connect the definition for the n^{th} term of C in the recursive definition with a general form of an exponential equation (non-recursive) such as $g(x) = a \cdot b^x$, where a represents the starting term and b represents the growth factor. Students should understand that since C is a geometric sequence, it is a type of exponential function that has a domain restricted to a subset of the integers. Therefore, their graphs in question 3 should consist only of unconnected dots.

RESPONSIVE STRATEGY

Use this Co-Craft Questions routine to help students consider the context of this problem. Show the first part of the task (through "And so on."), hiding the $\frac{1}{3}$s. Ask students to write their own mathematical questions that could be asked about the situation. They should look for whether or how the amount of remaining cake shows up in students' questions, and whether students assume that each person takes the same fraction of the cake. Once students compare their questions, reveal the $\frac{1}{3}$s and the activity's questions. For students struggling to get started, ask them to create a diagram to represent the situation, or have two students act out the situation (use a piece of paper to represent the cake, and repeatedly remove $\frac{1}{3}$ of it).

Co-Craft Questions (MLR5)

- Ask a student to share their graph or consider displaying a graph of the situation. Ask students:
  - "What is a reasonable domain of C?" (Without regard to the context, the domain of C is $n \geq 0$, where n is an integer.)
  - "Is there a limit to how many people can get cake? Does this impact the reasonable domain?" (There will eventually be too little cake to reasonably take $\frac{1}{3}$ of it. Deciding when there is too little cake left is up to the modeler but should be reflected in the reported domain. After 8 people take their cake, less than 5% of the cake is left, therefore, one could say the reasonable domain is integers from 0 to 10.)

RESPONSIVE STRATEGY

Represent the same information through different modalities by using physical objects to represent abstract concepts. Some students may benefit by creating a table of values for the non-recursive equations, in order to compare results with the table created using the recursive pattern.

Supports accessibility for: Conceptual processing; Visual-spatial processing

DO THE MATH

PLANNING NOTES

Lesson Debrief (*10 minutes*)

The purpose of this lesson is for students to model real-world situations using arithmetic and geometric sequences, and to consider an appropriate domain for those sequences.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

The goal of this discussion is to highlight different representations of sequences that have come up in this lesson and reflect on their use. The last question focuses student thinking on the limitations of graphing a recursively defined function. Consider helping students generalize their responses to the prompts drawing off of their work from this unit.

Use the following questions:

- Is one representation (graph, table, equation) easier to solve problems with than another? (Students might note that it depends on how the information is given and what question is being asked. For example, graphing the definition for the n^{th} term might be easier for determining whether and when population B ever overtakes population A. But the table might be easier for determining whether the sequences are arithmetic, geometric, or neither. Using a spreadsheet might be most efficient for both tasks.)
- How can you determine that a sequence is arithmetic or geometric from a
  - Recursive definition? (An arithmetic sequence will add a common difference to the previous term and a geometric definition will multiply the previous term by a common ratio.)

(*continued*)

PLANNING NOTES

- Explicit definition? (An arithmetic sequence will be defined by an explicit equation that can be written in the form of a linear function $y = mx + b$. A geometric sequence will be defined by an explicit equation that can be written in the form of an exponential function $y = a \cdot b^x$.)
- Graph? (The graph of a sequence should be discrete. An arithmetic sequence will be in the shape of a straight line and a geometric sequence will be in a curved line like the graph of an exponential function.)
- Table? (An arithmetic sequence will have a common difference from one term to the next. A geometric sequence will have a common ratio from one term to the next.)

- How can the context of a problem place limitations on the domain of a sequence? What effect does this have on the graph? (Geometric and arithmetic sequences are limited to the subset of integers since the domain consists of the term numbers. This implies that the graph of a sequence will be a discrete graph, often represented by a discrete set of points. Additionally, there may be some upper limit for the sequences. For example, in the cake-cutting problem, it would not be reasonable to go past about 10 cuts.)

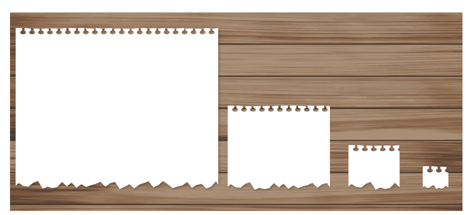
Some important connections to be made include:

- Arithmetic sequences always have a rate of change that can be interpreted in representations of the sequence.
- Graphs of arithmetic sequences appear linear, where the slope is the rate of change.
- The definition for the n^{th} term of an arithmetic sequence can be written like a linear equation.
- Geometric sequences always have a growth or decay factor that can be interpreted in representations of the sequence.
- Graphs of geometric sequences appear exponential and you can calculate the growth/decay factor and growth/decay rate given the coordinates of points on the graph.
- The definition for the n^{th} term of a geometric sequence can be written like an exponential equation.

Student Lesson Summary and Glossary

The model we use for a function can depend on what we want to do.

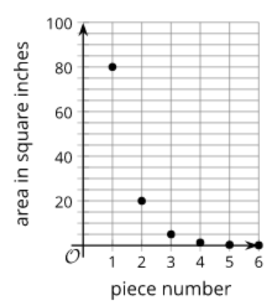
For example, an 8-inch-by-10-inch piece of paper has an area of 80 square inches. Picture a set of pieces of paper, each half the length and half the width of the previous piece.



Define the sequence A so that $A(n)$ is the area, in square inches, of the n^{th} piece. Each new area is $\frac{1}{4}$ the previous area, so we can define A recursively as:

$$A_1 = 80,\ A_n = A_{n-1} \cdot \frac{1}{4} \text{ for } n \geq 2$$

But for n-values larger than 5 or 6, the model isn't realistic since cutting a sheet of paper accurately when it is less than $\frac{1}{50}$ of a square inch isn't something we can do well with a pair of scissors. We can see this by looking at the graph of $y = A(n)$ shown here.



If we wanted to define the n^{th} term of A, it's helpful to first notice that the area of the n^{th} piece is given by $80 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot .... \cdot \frac{1}{4}$, where there are $n - 1$ factors of $\frac{1}{4}$. Then we can write

$$A(n) = 80 \cdot (\tfrac{1}{4})^{n-1},\ n \geq 1$$

We can use this definition to calculate a value of A without having to calculate all the ones that came before it. But since there are fewer than 10 values that make sense for A, since we can't cut very tiny pieces using scissors, in this situation we could just use the first definition we found to calculate different values of A.

Cool-down: Two Bacteria Populations (*5 minutes*)

Addressing: NC.M1.F-BF.2

Cool-down Guidance: Press Pause
Provide opportunities to support students in creating explicit and recursive definitions using practice problems prior to the End-of-Unit Assessment.

Cool-down

The table shows two bacteria populations changing over time, measured in hours since the populations were first counted.

1. Describe a pattern for how each population changes from one hour to the next.
2. Write an explicit definition for each population.
3. Write a recursive definition for each population.

Time (hours)	Population A (millions)	Population B (millions)
0	12	64
1	6	48
2	3	36
3	1.5	27

Student Reflection:

Using math to study population is a common practice. Do you think explicit or recursive definitions make more sense in studying populations of people over time? What additional information might you need?

DO THE MATH

INDIVIDUAL STUDENT DATA

SUMMARY DATA

NEXT STEPS

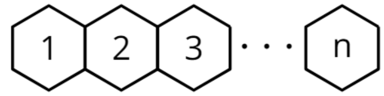
TEACHER REFLECTION

What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?

What question do you wish you had asked today? When and why should you have asked it?

Practice Problems

1. A party will have hexagonal tables placed together with space for one person on each open side.



   a. Complete this table showing the number of people $P(n)$ who can sit at n tables.

n	1	2	3	4	5
$P(n)$	6				

   b. Describe how the number of people who can sit at the tables changes with each step.

   c. Explain why $P(3.2)$ does not make sense in this scenario.

   d. Define P recursively and for the n^{th} term.

2. Diego is making a stack of pennies. He starts with five pennies and then adds one penny at a time. A penny is 1.52 mm thick.

n	$h(n)$
0	7.6
1	
2	
3	

   a. Complete the table with the height of the stack $h(n)$, in mm, after n pennies have been added.

   b. Does $h(1.52)$ make sense? Explain how you know.

3. A piece of paper has an area of 80 square inches. A person cuts off $\frac{1}{4}$ of the piece of paper. Then a second person cuts off $\frac{1}{4}$ of the remaining paper. A third person cuts off $\frac{1}{4}$ what is left, and so on.

n	$A(n)$
0	80
1	
2	
3	

   a. Complete the table where $A(n)$ is the area, in square inches, of the remaining paper after the n^{th} person cuts off their fraction.

   b. Define A for the n^{th} term.

   c. What is a reasonable domain for the function A? Explain how you know.

4. A piece of paper is 0.05 mm thick.

n	$t(n)$
0	0.05
1	
2	
3	

   a. Complete the table with the thickness of the paper $t(n)$, in mm, after it has been folded n times.

   b. Does $t(0.5)$ make sense? Explain how you know.

5. A piece of paper has an area of 96 square inches.

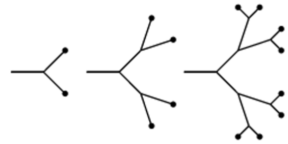
n	$A(n)$
0	96
1	
2	
3	

   a. Complete the table with the area of the piece of paper $A(n)$, in square inches, after it is folded in half n times.

   b. Define A for the n^{th} term.

   c. What is a reasonable domain for the function A? Explain how you know.

6. Here is a growing pattern:



   a. Describe how the number of dots increases from stage 1 to stage 3.
   b. Write a definition for sequence D, so that $D(n)$ is the number of dots in stage n.
   c. Is D a geometric sequence, an arithmetic sequence, or neither? Explain how you know.

7. A paper clip weighs 0.5 grams, and an empty envelope weighs 6.75 grams.
   a. Han adds paper clips one at a time to an empty envelope. Complete the table with the weight of the envelope, $w(n)$, in grams, after n paper clips have been added.
   b. Does $w(10.25)$ make sense? Explain how you know.

n	$w(n)$
0	6.75
1	
2	
3	

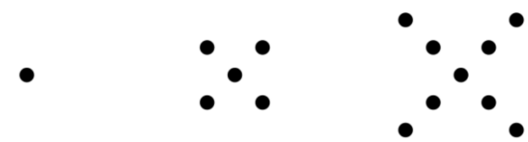
8.
   a. An arithmetic sequence has $a(1) = 4$ and $a(2) = 16$. Explain or show how to find the value of $a(15)$.
   b. A geometric sequence has $g(0) = 4$ and $g(1) = 16$. Explain or show how to find the value of $g(15)$.

   (From Unit 8, Lesson 6)

9. An arithmetic sequence k starts 4, 13, ….Explain how you would calculate the value of the 5,000th term.

   (From Unit 8, Lesson 6)

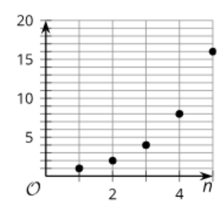
10.



   a. Describe the pattern that you see in the sequence of the figures above.[1]
   b. Assuming the pattern continues in the same way, how many dots are there at 3 minutes?
   c. How many dots are there at 100 minutes?
   d. How many dots are there at t minutes? Solve the problem by your preferred method. Your solution should indicate how many dots will be in the pattern at 3 minutes, 100 minutes, and t minutes. Be sure to show how your solution relates to the picture and how you arrived at your solution.

   (From Unit 8, Lesson 6)

11. Here is a graph of sequence t. Define t recursively using function notation.



   (From Unit 8, Lesson 5)

12. Match each sequence with one of the recursive definitions. Note that only the part of the definition showing the relationship between the current term and the previous term is given so as not to give away the solutions. One of the sequences matches two recursive definitions.

   a. $a(n) = a(n-1) - 4$
   b. $b(n) = b(n-1) + 0$
   c. $c(n) = -\frac{1}{2} \cdot c(n-1)$
   d. $d(n) = 1 \cdot d(n-1)$

   1. 7, 3, -1, -5
   2. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$
   3. 8, 8, 8, 8

   (From Unit 8, Lesson 4)

[1] Adapted from Open Up Resources. Access the full curriculum, supporting tools, and educator communities at openupresources.org.

Lessons 8 & 9: Mathematical Modeling[1]

PREPARATION

Lesson Goals	Learning Targets
• Use mathematical tools to represent, interpret, analyze, generalize, communicate about, and make predictions about how real-world quantities vary in relation to each other. • Use observational and experimental data to test, improve, and validate mathematical models.	• I can use mathematics to model real-world situations. • I can test and improve mathematical models for accuracy in representing and predicting real things.

Lesson Narrative

Modeling is the link between the mathematics students learn in school and the problems they will face in college, career, and life. Time spent on modeling is crucial, as it prepares students to use math to handle technical subjects in their further studies, and problem solve and make decisions that adults regularly encounter in their lives.

For Lessons 8 and 9, there are eight choices of modeling prompts: Modeling Prompts #1 and #2 are available in Unit 2; Modeling Prompts #3 and #4 are available in Unit 4; Modeling Prompts #5 and #6 are available in Unit 6; and Modeling Prompts #7 and #8 are provided here.

Remind students what modeling is and what is expected of them as a modeler using the following resources and guidance:

Organizing Principles about Mathematical Modeling

- The purpose of mathematical modeling in school mathematics courses is for students to understand that they can use math to better understand things in the world that interest them.
- Mathematical modeling is different from solving word problems. It often feels like initially there is not enough information to answer the question. There should be room to interpret the problem. There ought to be a range of acceptable assumptions and answers. Modeling requires genuine choices to be made by the modeler.
- It is expected that students have support from their teacher and classmates while modeling with mathematics. It is not a solitary activity. Assessment should focus on feedback that helps students improve their modeling skills.

Things the Modeler Does When Modeling with Mathematics (NGA 2010)

1. *Pose a problem* that can be explored with quantitative methods. Identify variables in the situation and select those that represent essential features.
2. *Formulate a model*: Create and select geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between variables.
3. *Compute*: Analyze these relationships and perform computations to draw conclusions.
4. *Interpret* the conclusions in terms of the original situation.
5. *Validate* the conclusions by comparing them with the situation. Iterate if necessary to improve the model.
6. *Report* the conclusions and the reasoning behind them.

It's important to recognize that in practice, these actions don't often happen in a nice, neat order.

[1] Adapted from IM 9–12 Math Algebra 1 Modeling Prompts https://curriculum.illustrativemathematics.org/HS/index.html,

Preparing for a Modeling Prompt

Ideas for Setting Up an Environment Conducive to Modeling

- Provide plenty of blank whiteboard or chalkboard space for groups to work together comfortably. “Vertical non-permanent surfaces” are most conducive to productive collaborative work. “Vertical” means on a vertical wall, which is better than horizontally on a tabletop, and “non-permanent” means something like a dry erase board, which is better than something like chart paper (Liljedahl 2016).
- Ensure that students have easy access to any tools that might be useful for the task. These might include:
  - supply table containing geometry tools, calculators, scratch paper, graph paper, dry erase markers (ideally a different color for each group member), post-its
  - electronic devices to access digital tools (like graphing technology, dynamic geometry software, or statistical technology)
- Think about how to help students manage the time that is available to work on the task. For example:
  - Display a countdown timer for intermittent points in the lesson when groups are asked to summarize their progress.
  - Decide what time to ask groups to transition to writing down their findings in a somewhat organized way (perhaps 15 minutes before the end of the class).

How to Prepare and Conduct the Modeling Lesson

- Decide which version of the prompt students will receive, based on the lift-analysis, timing, and access to data.
- Have data ready to share if planning to give it when students ask.
- Decide if students will be offered a template for organizing modeling work.
- Decide to what extent students are expected to iterate and refine their model. The amount of time available can influence how much time students have to refine their model. If time is short, students may not engage as much in that part of the modeling cycle. WIth more time, it is more reasonable to expect students to iterate and refine their model once or even several times.
- Decide how students will report their results. Again, if time is short, this may be a rough visual display on a whiteboard. If more time is available, students might create a more formal report, slideshow, blog post, poster, mockup of an artifact like a letter to a specific audience, smartphone app, menu, or set of policies for a government entity to consider. One way to scaffold this work is to ask students to turn in a certain number of presentation slides: one that states the assumptions made, one that describes the model, and one or more slides with their conclusions or recommendations.
- Develop task-specific “look-fors” for each dimension of the provided rubric. What do you anticipate and hope to see in student work?

Organizing Students Into Teams or Groups

- Mathematical modeling is not a solitary activity. It works best when students have support from each other and their teacher.
- Working with a team can make it possible to complete the work in a finite amount of class time. For example, the team may decide it wants to vary one element of the prompt and compute the output for each variation. What would be many tedious calculations for one person could be only a few calculations for each team member.
- The members of good modeling groups bring a diverse set of skills and points of view. Create and share a Multiple Abilities List with students.
- Scramble the members of modeling teams often, so that students have opportunities to play different roles.

Ways to Support Students While They Work on a Modeling Prompt

- Coach students on ways to organize their work.
- Provide a template to help students organize their thinking. Over time, some groups may transition away from needing to use a template.
- Engage students in the *Three Reads* instructional routine to ensure comprehension of the prompt.
- Remind students of the variety of tools that are available to them.
- If students get stuck or run out of ideas, help move them forward with a question that prompts them to focus on a specific part of the modeling cycle. For example:
  - “What quantities are important? Which ones change and which ones stay the same?”
  - “What information do you know? What information would it be nice to know? How could you get that information? What reasonable assumption could you make?”
  - “What pictures, diagrams, graphs, or equations might help people understand the relationships between the quantities?”
  - “How are you describing the situation mathematically? Where does your solution come from?”
  - “Under what conditions does your model work? When might it not work?”
  - “How could you make your model better? How could you make your model more useful under more conditions?”
  - “What parts of your solution might be confusing to someone reading it? How could you make it more clear?”

What aspects of modeling have students in your class done well with this year? What aspects of modeling will you focus on in your support during these lessons?

Agenda, Materials, and Preparation

- How to Interpret the Provided Lift Analysis, Modeling Advice, and Modeling Rubric blackline masters
- **Modeling Prompt #7: Planning a Vacation**
  - Modeling Prompt #7 (print 1 copy per student)
- **Modeling Prompt #8: Planning a Concert**
  - Modeling Prompt #8 (print 1 copy per student)

LESSON

Modeling Prompt #7: Planning a Vacation

In this modeling prompt, students will work on planning a vacation. There are two versions of this prompt: 7A and 7B. In 7A, students determine who is going on the vacation, the budget for the trip, and what expenses will need to be considered. During their presentation, students using prompt 7A create a visual display for someone who wants to take a similar vacation on a different budget. In 7B, students are provided the number of people going on the vacation, the budget, and some suggested categories of expenses. In both versions, students will need to choose where the vacation is taking place, do research, and make and name assumptions. Determine, in advance, which Modeling Prompt (7A or 7B) students will receive, based on the lift-analysis, timing, and access to data.

Note: to take the focus off of spending money, instead of planning a vacation, this modeling task could be adapted to having students plan a backcountry hiking trip where they consider things like:

- Where do you want to go?
- For how long? How far can you hike each day?
- How much do you pack, and what are the weight restrictions?
- How much food do you need, and how much can you carry?

Student Task Statement 7A Lift Analysis

Attribute	DQ	QI	SD	AD	M	Avg
Lift	1	1	2	2	2	1.6

Student Task Statement 7B Lift Analysis

Attribute	DQ	QI	SD	AD	M	Avg
Lift	1	0	1	2	2	1.2

Step 1 (*Optional; review materials as necessary*)

- Display the Advice on Modeling and Modeling Rubric blackline masters.
- Facilitate a discussion around modeling. Share some of the following ideas:
  - Modeling prompts are often expressed in words, but unlike word problems, modeling prompts challenge the modeler to make reasonable assumptions, decide what information is important, ask or research for more information if needed, think creatively within constraints, and consider the implications of the model.
  - The process of modeling is cyclical, and it does not end by producing a "correct answer."
  - A mathematical model expresses a simplified relationship in the real world; models can be rough but still useful.
  - Models can often be refined to represent the real-world relationship more accurately.
  - Responses to modeling prompts can vary widely, but they often contain certain pieces: assumptions, calculations, a mathematical model (stated with an equation or equations, with a graph, with a geometric diagram, or in words), conclusions, and generalizations.
- Provide time for students to ask clarifying questions. Students should have a good-enough understanding of the rubric, however, deep understanding of the rubric is not needed at this time.

Step 2

- Pass out the pre-determined appropriate blackline master Modeling Prompt #7 (7A or 7B).
- Students can be arranged in groups in advance, they can choose groups, or groups can be determined by using visibly random grouping.
- Tell students that they have complete freedom to decide what type of trip they want to plan such as a plane trip, car trip, cruise, or hiking trip. Reinforce that they have to keep track of all their assumptions and account for all expenses.

> M **Monitoring Tip:** Monitor for students who try to plan something too easy, such as an all-inclusive vacation or a "staycation" where all expenses can be spent on food and entertainment. Ask them to come up with a secondary plan that requires more complicated travel plans (for example, the cruise gets canceled due to bad weather, or their dwelling is being remodeled and they have to leave).

Modeling Prompt 7A

1. You are planning a vacation. Start by deciding:
   a. Who is going on this trip?
   b. What type of vacation are you taking?
   c. What is your budget? To determine your budget, find the median income for your area and allocate either 5%, 8%, or 12% towards your vacation budget.
2. Make a detailed plan outlining the logistics and all expenses for the vacation.
3. Now, analyze your plan. Categorize the different expenses of your trip.
   a. What percentage of your budget are you spending on different categories?
   b. Which of the costs are one-time costs, and which costs depend on the length of the vacation or on the number of people going on vacation?
   c. Describe how the total cost changes if you change the length of the trip or the number of people.
4. Prepare a presentation about your vacation. Include a visual display that would help somebody who wants to take a similar vacation on a different budget decide how long their vacation could be.

Modeling Prompt 7B

You are planning a vacation for a family of four. Your budget for the trip is $3,500. This must cover:

- transportation
- lodging (like hotels or camp sites)
- food
- entertainment

1. Make a detailed plan with the activities and expenses for the trip.
2. Now, analyze your vacation. Categorize the different expenses of your trip.
    a. What percentage of your budget are you spending on different categories?
    b. Which of the costs are one-time costs, and which costs depend on the length of the vacation?
    c. What other quantities could be considered variables?
    d. Describe how the total cost changes if you increase the length of the trip.
3. Prepare a presentation about your trip and all expenses. Include what percentages of the total budget are spent in the different categories. Describe mathematically how the cost of the vacation changes if you increase the length of the trip.

Step 3

- Consider building in at least one structured opportunity for groups to iterate their model (for example, a “dry run” or a “first draft” presentation to a tiny audience). The job of the audience is to poke holes in the vacation plan, asking things like, “How do you plan to get from the airport to the hotel?”; “What are you going to eat on your travel days?”; or “That expense seems unreasonably low; how did you research it?” Then, the presenter has time to incorporate the feedback into their final draft.
- Ask students to check in with you once they have outlined their plan. They should consider the cost of transportation, food, lodging, and entertainment, at a minimum, and have a plan to find realistic expenses for each category.

Step 4

- Remind students that modeling is a cycle, and they should evaluate their own models and then refine them as necessary.
- After sufficient work time, each group or pair should share their solutions with the class. These prompts suggest a presentation; however, students could also share by: creating a visual that can be observed doing a gallery walk, creating a slide deck, uploading a scan or photo of their work to a shared online space, or by any method that works best for the class.

Step 5

- Provide students time to reflect on their experience with this modeling prompt in their Student Workbook.

Modeling Prompt #8: Planning a Concert

In this modeling prompt, students will work on helping a charity plan a concert to raise money. There are two versions of this prompt: 8A and 8B. In both versions, students will determine the location of the concert and how much tickets should cost. In 8B, students are given guidance on how to use the research data to decide on the cost. Determine, in advance, which Modeling Prompt (8A or 8B) students will receive, based on the lift-analysis, timing, and access to data.

The amount of detail students can put into their plan is limited only by the amount of research they are willing to do. To find a place to have the concert, students can use search terms like "rent event space"; there are websites that will show spaces for rent nearby. If students have trouble finding local spaces, they can pretend the concert is in the nearest large city instead.

Student Task Statement 8A Lift Analysis

Attribute	DQ	QI	SD	AD	M	Avg
Lift	2	1	2	2	2	1.8

Student Task Statement 8B Lift Analysis

Attribute	DQ	QI	SD	AD	M	Avg
Lift	2	1	2	2	1	1.6

Step 1 (*Optional; review materials as necessary*)

- Display the Advice on Modeling and Modeling Rubric blackline masters.
- Facilitate a discussion around modeling.
  - See discussion suggestions above in Step 1 of Modeling Prompt 7.
- Provide time for students to ask clarifying questions. Students should have a good-enough understanding of the rubric, however, deep understanding of the rubric is not needed at this time.

Step 2

- Pass out the pre-determined appropriate blackline master Modeling Prompt #8 (8A or 8B).
  - **Modeling Prompt 8A**

    ***Advancing Student Thinking*:** For prompt 8A, students will need to think carefully about the survey results in order to use them to decide on a ticket price. A very helpful first step is to figure out how many of the people surveyed would buy a ticket for $10, how many would buy a ticket for $20, and so on. Students should remember that if someone says, for example, that they would pay at most $30, this means they would also buy a ticket if it were less than $30. If students graph the number of people who would buy a ticket for each price, they will be able to see how the ticket price affects the number of tickets bought, and they can find a linear equation that describes how the number of tickets depends on the price. This will help them create an equation for the amount of money the charity will make by selling tickets at each price, as shown in the sample response.

  - **Modeling Prompt 8B**

    ***Advancing Student Thinking*:** Although prompt 8B gives students some guidance about how to decide on a ticket price using the survey information, they have not done this sort of work before, and they may struggle. If they are unsure how to begin or how the hints are supposed to help, try asking them to look at problems they have seen before that involve equations for income or profit. These equations can all be factored into the form $x(-ax+b)$. Ask students how this relates to the fact that $amount\ of\ money\ made = price \cdot number\ of\ things\ sold$. (The price is x. The number of things sold is a linear equation that depends on x, because the more something costs, the fewer people buy it.)

- Students can be arranged in groups in advance, they can choose groups, or groups can be determined by using visibly random grouping.
- Ask students if they have been to a concert or a similar event before. Invite students to share what the experience was like. For example, where was the event? Were a lot of other people there? Who was performing?

- Ask students what they would need to think about if they were planning a concert. After some quiet think time, ask students to share their ideas with their group. Then invite students to share something they or someone in their group thought of, and record the ideas for all to see. Possible responses include:

  - which band or performer to have
  - how much to charge for tickets
  - whether to charge for parking
  - how many people to expect
  - where to have the concert
  - what kinds of food and drink to sell
  - whether to sell backstage passes or other special add-ons

  For each item that students suggest, ask whether it represents something that will cost the organizers money or that will make the organizers money. Some items may do both—for example, the organizers will need to pay the band, but the band is also the reason the audience is paying money for tickets. In this case, the band will probably bring in more money than they are paid, so overall paying the band is not a cost. Remind students that "profit" is the amount the organizers make minus the amount they pay. Tell students that in this task they will plan a concert themselves.

Modeling Prompt 8A

1. A charity is going to raise money by having a concert. You are helping to plan the concert. You will need to decide where the concert should be and how much the tickets should cost. You will also need to predict how much profit the charity will make from the concert.

   The charity has done some research about ticket prices. They chose 100 people at random and asked them, "What is the most that you would pay for a ticket to a charity concert?" The responses are shown in the table.

Ticket price ($)	Number of people who would pay at most this much
10	15
20	22
30	13
40	15
50	17
60	15
65	3

   Here are some questions to guide you as you make your plan:

   - Based on the information from the survey, how much should tickets cost?
   - What kind of performer do you want to have at the concert? A local band? A famous singer? An orchestra? A "battle of the bands" featuring many different groups? More famous performers may draw a larger crowd, but they may also need to be paid more. Do research to find out what a fair amount of money would be.
   - Research some possible concert venues. How much would they cost to rent, and how many people can they hold?
   - If the tickets are sold at the price you recommend, which venue will create the most profit?
   - Should the charity also sell other things at the concert, like food or T-shirts? If you think so, you can also recommend this to the charity and predict how much profit they'll make.

2. Create a presentation to explain your plan and your reasoning to the charity's directors. Include an estimate of the costs and profit.

Modeling Prompt 8B

1. A charity is going to raise money by having a concert. You are helping to plan the concert. You will need to decide where the concert should be and how much the tickets should cost. You will also need to predict how much profit the charity will make from the concert.

   The charity has done some research about ticket prices. They chose 100 people at random and asked them, “What is the most that you would pay for a ticket to a charity concert?” The responses are shown in the table.

Ticket price ($)	Number of people who would pay at most this much
10	15
20	22
30	13
40	15
50	17
60	15
65	3

   You can use this information to find out which ticket price will bring in the most money. Here is how:

   - First, figure out how many people would buy a ticket for $10, how many would buy a ticket for $20, and so on.
   - Then create a graph with the ticket price as the independent variable and the number of people who buy tickets as the dependent variable. Plot the points you found in the first step, and then find a linear equation that shows how the number of people depends on the ticket price.
   - The amount of money that the charity will make from ticket sales is the ticket price multiplied by the number of people who buy tickets. Use the linear equation you found to create a quadratic equation that shows how the price of tickets affects the amount of money the charity will make from the tickets.

   After you have decided on a ticket price, here are some questions to think about:

   - What kind of performer do you want to have at the concert? A local band? A famous singer? An orchestra? A “battle of the bands” featuring many different groups? More famous performers may draw a larger crowd, but they may also need to be paid more. Do research to find out what a fair amount of money would be.
   - Research some possible concert venues. How much would they cost to rent, and how many people can they hold?
   - If the tickets are sold at the price you recommend, which venue will create the most profit?
   - Should the charity also sell other things at the concert, like food or T-shirts? If you think so, you can also recommend this to the charity and predict how much profit they’ll make.

2. Create a presentation to explain your plan and your reasoning to the charity’s directors. Include an estimate of the costs and profit.

Step 3

- Remind students that modeling is a cycle, and they should evaluate their own models and then refine them as necessary.
- After sufficient work time, each group or pair should share their solutions with the class. These prompts suggest a presentation, however students could also share by: creating a visual that can be observed doing a gallery walk, creating a slide deck, uploading a scan or photo of their work to a shared online space, or by any method that works best for the class.

Step 4

- Provide students time to reflect on their experience with this modeling prompt in their Student Workbook.

TEACHER REFLECTION

Which modeling prompt did you choose for this lesson? What version of the prompt did students receive? How did you make these decisions? Would you make a different choice next time you facilitate this lesson?

In what ways have students improved in their engagement with and solving of modeling prompts this year/semester?

In what ways have you improved in facilitation of modeling prompts?

Lesson 10: Post-Test Activities

PREPARATION

Lesson Goal	Learning Targets
• Provide students the opportunity to reflect and share feedback on their own progress and on the culture and instruction happening in the class.	• I can reflect on my progress in mathematics. • I can share feedback that can help make me and my teacher grow.

Lesson Narrative

This lesson, which should occur after the Unit 8 End-of-Unit Assessment, allows for students to reflect on the unit and the course, share feedback, and conference with the teacher.

Gathering student feedback is a powerful and strategic way to learn about students and improve instructional practices. It also creates student and family buy-in and centers students as decision makers and problem solvers in their own learning.

What feedback are you anticipating receiving from your students?

Agenda, Materials, and Preparation

- **Activity 1** (*45 minutes*)
  - Student Course Survey (print 1 copy per student)
- **Activity 2** (*Optional*)
  - Alphabetical Advice (print 1 copy per student)
  - Materials for “thank you” notes
  - Materials for writing letters to themselves

LESSON

Activity 1: Student Course Survey (*45 minutes*)

The Student Course Survey is a critical opportunity for teachers to gather low-stakes, non-evaluative feedback to support equitable instruction. The survey is also highly beneficial for students as it is designed to encourage self-awareness, self-management, social awareness, relationship skills, and responsible decision making. Provide students a chance to quietly and independently complete this survey after they complete their testing.

One-on-One Conferences

Conducting one-on-one conferences with students at the conclusion of the course is recommended. Potential topics include:

- student responses to the daily student reflections
- student response to the end-of-course student survey (as students finish them)
- reflecting on ways the student has grown mathematically across the course
- goal setting for Math 2

Activity 2: Additional Activities (*Optional*)

If students complete their surveys early or there is additional time, here are some potential optional activities.

- Alphabetical Advice:[1] Students share advice for a future Math 1 class starting with each letter of the alphabet. Compile the best into one document to share with your students next semester.
- Gratitude: Students write a "thank you" note to at least one student in this class who supported their learning in positive ways. Ask them to include specific details about what or how they worked with you collaboratively that was helpful.
- Looking Ahead: Students write themselves a letter citing what they are proud of from Math 1 and naming what they want to continue and how they want to grow in Math 2. Ideally, provide a written response to each student for students to keep.

TEACHER REFLECTION

Name three moments from this year that you are particularly proud of as a teacher. Why do those moments jump out?

What information from the student surveys was most exciting? What was difficult to read? What are some ways you will use what you learned from students next year/semester?

[1] Adapted from Sarah Carter